The generalized Non-Linear Helmholtz (gNLH) equation can be used to model broad spatial beams propagating at arbitrarily large angles, relative to the reference direction, in planar-waveguide geometry. The physical equivalence of transverse and longitudinal dimensions is preserved, and new effects are predicted which have no counterpart in paraxial theory. Exact analytical soliton solutions and conserved quantities of the gNLH equation are presented. Well-tested numerical perturbative techniques examine the role of the new solutions as robust attractors in the system dynamics.

**gNLH Equation & Soliton Solutions**

The propagation of spatial beams in planar waveguides is routinely described by models based on the Non-Linear Schrödinger (NLS) equation. Such equations are constrained by the paraxial approximation, making them inadequate for studying some regimes of physical interest. When the dominant type of non-paraxiality in the system arises solely from the oblique propagation of broad beams, an accurate model requires the restoration of $\mathcal{O}(k^2)$ in the governing wave equation, rather than higher-order transverse correction terms [4]. The gNLH equation has been derived to describe the evolution of beams propagating at arbitrary angles with respect to the reference direction, as well as to include non-linear refractive-index effects from the atmospheric $\kappa$-effect (proportional to intensity) responses [4-5].

$$\frac{\partial \psi}{\partial z} = -i \left[ \frac{1}{2} \left( 1 + \frac{1}{2} \kappa |\psi|^2 \right) \psi + \frac{1}{2} \kappa_0 |\psi|^2 \psi \right] + i \kappa_0 \frac{\partial^2 |\psi|^2}{\partial x^2} \psi$$

This equation admits exact analytical soliton solutions that exhibit non-trivial corrections to their paraxial counterparts [4].

$$\psi(x,z) = \sqrt{\frac{N_0}{4 \kappa_0 \kappa}} \exp \left( -i \frac{1}{2} \kappa_0 \frac{1}{2} \kappa |\psi|^2 \right) \exp \left( -i \frac{1}{2} \kappa_0 \frac{1}{2} \kappa |\psi|^2 \right) \exp \left( -i \frac{1}{2} \kappa_0 \frac{1}{2} \kappa |\psi|^2 \right)$$

This solution represents a spatial beam propagating at a non-trivial angle $\theta$ with respect to the reference direction.

The Helmholtz correction factor $2\kappa_0^2$ can be arbitrarily large, even for broad beams. This regime defines a Helmholtz-type of non-paraxiality.

**Soliton Stability**

When $\rho = 0$, this equation describes the stability of solitons generated by the paraxial approximation. An examination of the system's stability using the method of characteristics shows that the system's stability is dependent on the parameters $\kappa$ and $\kappa_0$. In the Helmholtz case, there is no known (mathematically rigorous) analog of this stability result for the analytical solutions obtained by the $\kappa_0$-type. The analytical determination of stability remains open, and is currently a purely numerical pursuit. The results of this analysis are presented [6].

Conserved Quantities

Using Lagrangian field-theoretic techniques, three conserved quantities of the gNLH equation can be identified. The energy $E$, the momentum $P$, and the Hamiltonian $H$ of the system. The gNLH model is regarded as the Euler-Lagrange equation of motion for a Lagrangian density, and a pair of canonically-conjugate momentum variables are defined. Noether’s theorem can then be exploited to derive the system equations.

$$M = \int \left( \frac{\partial L}{\partial \psi} - \frac{\partial L}{\partial \psi^*} \right) d^2x, \quad H = \int \left( \frac{\partial L}{\partial \psi} - \frac{\partial L}{\partial \psi^*} \right) d^2x$$

The integral representation of the invariants is vital for monitoring the integrity of any numerical scheme used to solve the gNLH equation [6].

**Helmholtz Solitons**

A weakly-localized non-linear wave of the gNLH equation is the **Helmholtz soliton**. This solution has much lower power (e.g., Lorentzian) asymmetry relative to the strongly-localized soliton-type (exponential) solutions. The solitons are supported only in the Type I competing regime where, in the limit $N_0 \rightarrow 0$, there remains a non-zero energy-flux, $(H_{NLH} > 0)$.

$$\psi(x,z) = \sqrt{\frac{N_0}{4 \kappa_0 \kappa}} \exp \left( -i \frac{1}{2} \kappa_0 \frac{1}{2} \kappa |\psi|^2 \right) \exp \left( -i \frac{1}{2} \kappa_0 \frac{1}{2} \kappa |\psi|^2 \right) \exp \left( -i \frac{1}{2} \kappa_0 \frac{1}{2} \kappa |\psi|^2 \right)$$

The Lorentzian asymmetries are seen from $J_{NLH} = \int |\psi|^2 \exp(-i\kappa_0 |\psi|^2) \exp(-i\kappa |\psi|^2) \exp(-i\kappa_0 |\psi|^2) d^2x$.

Algebraic solitary waves are a common feature of nonlinear systems, occurring, for example, in fluid mechanics. It could be predicted a-priori that paraxial wave optics, governed by NLS-type equations, must also support algebraic solitons from the fluid mechanics–nonlinear optics analog. Such waves do exist and have been reported by several authors, such as in [5]. However, this is the first known reporting of algebraic solitons in fully 2-D nonparaxial Helmholtz-type systems.

**Recovery of Paraxial Solutions**

Known solutions of the paraxial model [5] corresponding to the gNLH equation can be recovered from the full Helmholtz solutions when an appropriate multiple limit is enforced.

**Solitons as Robust Attractors**

The numerical perturbative approach involves using initial conditions that correspond to exact solutions of the paraxial equation with transverse velocity $v_x$. For quasi-paraxial beams (where the first two conditions in the paraxial limit are met), rotational symmetry establishes a connection between $x$ and $y$. Examination of the beam along its propagation axis shows that the evolution is equivalent to that of an on-axis soliton whose width has been reduced by a factor of $(1 + \lambda)^{-1}$, where $\lambda = 1/(v_x/v_y)$. The figures below depict beam self-focusing solutions, depending upon the parameters involved $(\rho, \kappa, \kappa_0)$, a propagation and soliton regime: (i) soliton collision, where the total energy is conserved, and the collision is asymptotically similar to the initial conditions. The corresponding Helmholtz soliton is then classified as a stable fixed point. Alternatively, the oscillations may persist in the long term. In this case, the solution is a new type of stable limit cycle solution.

**Conclusions**

- The gNLH equation possesses exact analytical soliton solutions. Two distinct solution families exhibit strong (exponential) weak (power-law) localization of the beam energy.
- gNLH solitons are of intrinsic mathematical interest. They represent a novel contribution to the knowledge of soliton dynamics in fully 2-D non-integrable models. New algebraic solitons have been derived.
- Experimentally, the gNLH equation pertains directly to known materials, such as some semiconductor-doped glasses [2] and non-linear polymers [3]. This is an appropriate model for describing optical phenomena in broad transverse beams.
- gNLH solitons have been shown to be stable by limiting dynamics. As threshold is approached, this stability is lost and the initial condition undergoes destructive spreading. (c) Parabolic theory predicts this type of instability when the on-axis longitudinal gradient is negative.

**References**