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Hybrid block replacement and inspection policies for a multi-component system with heterogeneous component lives

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Abstract: Novel replacement policies that are hybrids of inspection maintenance and block replacement are developed for an n component series system in which the component parts used at successive replacements arise from a heterogeneous population. The heterogeneous nature of components implies a mixed distribution for time to failure. In these circumstances, a hybrid policy comprising two phases, an early inspection phase and a later wear-out replacement phase, may be appropriate. The policy has some similarity to burn-in maintenance. The simplest policy described is such a hybrid and comprises a block-type or periodic replacement policy with an embedded block or periodic inspection policy. We use a three state failure model, in which a component may be good, defective or failed, in order to consider inspection maintenance. Hybrid block replacement and age based inspection and opportunistic hybrid policies will also arise naturally in these circumstances and these are briefly investigated. For the simplest policy, an approximation is used to determine the long run cost and the system reliability. The policies have the interesting property that the system reliability may be a maximum when the long-run cost is close to its minimum. The failure model implies that the effect of maintenance is heterogeneous. The policies themselves imply that maintenance is carried out more prudently on newer than on older systems. The maintenance of traction motor bearings on underground trains is used to illustrate the ideas in the paper.

Keywords: block replacement, age based replacement, multi-component system, maintenance, inspection, mixtures.

1. Introduction

Maintenance policies that have an inspection element and a time based replacement element are investigated in this paper. The novelty of the paper lies with the fact that it considers such hybrid policies in the context of a multi-component system with heterogeneous components. Components are assumed to arise from a population that is a mixture of weak and strong components. In these circumstances, policies that comprise an initial inspection phase and a subsequent non-inspection phase that culminates in replacement may have some merit. The purpose of the inspection phase is to prevent early failures due to weak components. Burn-in maintenance [1] has a similar purpose. Scarf et al. [2] investigated a hybrid age based replacement and inspection policy for such a system, albeit in the single component case. If this policy is applied to a system comprising n components in series, then maintenance of individual components on an age basis will go out of phase. A block-type policy then becomes a sensible alternative, and in realistic scenarios will

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nearly always be cheaper than individual (or collective) age-based policies for the components. The combination of preventive replacements for a multi-component system when the replacements have been determined component-wise is an interesting problem [3], and might be explored when components are heterogeneous, but this is another matter.

A hybrid block type policy would replace on failure and preventively replace periodically, and inspect on a periodic basis during an inspection phase. We might also consider a policy in which preventive replacements are time based (block-type) but inspections are age-based. In this latter policy, when a defective component is replaced at inspection, or a component is replaced on failure, the inspections of the replaced component (or all components) are rescheduled from the time of (defective) preventive replacement, but the subsequent preventive replacement of the system is not rescheduled. In practice, when there are multiple components in a system, the rescheduling of the inspections following preventive replacement of a defective item would not cause inspections to become unsynchronized—it would just mean some components would continue to be inspected for longer. Although, one could imagine a policy in which if one component is preventively replaced at inspection, the inspections of all items would be rescheduled. However, failure replacements would cause inspections to become unsynchronized. It soon becomes apparent that a multitude of policies are possible. We focus on a small number of policies. The choice of these policies is motivated to an extent by the context of case data relating to maintenance of traction motor bearing on underground trains—eight motors per train. Policies are compared on the basis of cost (long run cost per unit time), and reliability of operational function (mean time between operational failures). Because of the nature of the mixed failure model, these criteria may not be in conflict—that is, circumstances may exist for which both the (long run) cost is minimized and the mean time between operational failures is maximized.

In the next section, we describe the structure of the system, the model of the failure process, and the basic costs. Hybrid block replacement and inspection policies are then outlined in section 3. In section 4 we attempt to make some analytical progress with the analysis of the cost and reliability of the simplest policy, the block replacement and block inspection. Section 5 considers the other more complex policies. Case study data is used as the basis for a numerical study of various policies in section 6. A discussion of limitations and remaining issues then follows.

2. The system and failure model characteristics

The system comprises n statistically independent components in series. If any 1 component fails then a failure of the operational function of the system occurs. Failures of operational function of the system are undesirable and incur cost c_F , in addition to the cost of the replacement of the failed component (c_R) and the replacement set-up cost (c_S). Failed components are immediately apparent, and are replaced instantaneously. The system is renewed if all components are replaced. Component i is renewed on replacement of component i . The purpose of maintenance is two fold, to reduce costs and to prevent failure of operational function.

We suppose that a defect arises prior to failure, that defects are detectable at inspection, and that components which are defective on inspection are immediately replaced. The cost of inspection has a similar structure to the replacement cost: the set-up cost is c_S , the inspection cost

for each component is c_I , and the cost of replacement of a defective component is c_R . The time lapse between defect arrival and failure due to this defect is the delay time [4], denoted by Y . This assumption enables inspections to be modelled; an inspection can only be effective if the delay time is non-zero.

As in [2], we model the distribution of time to defect arrival, X , as a mixture: $f_X(x) = pf_1(x) + (1-p)f_2(x)$, where $f_1(x)$ is density of the time to defect arrival of the weak components and $f_2(x)$ likewise for the strong components; p is the mixing parameter. We use Weibull distributions for $f_1(x)$ and $f_2(x)$ although others would do here. Mixtures of this kind do not necessarily have an increasing failure (hazard) rate function, and a bathtub shaped hazard [5] is one possible form. Jiang and Murthy [6] have considered the use of these distributions in reliability studies in detail. The delay time is also modelled as a random variable. This could itself be a mixture, but to keep the parameter space within reasonable limits, we assume a simple exponential. By way of extension, it would be possible to model imperfect inspection by assuming that a proportion of defects have zero delay time, using a mixed exponential distribution [7]. The distribution of time to failure, Z , will then be a convolution of the defect arrival distribution and the delay time distribution.

3. Hybrid block replacement and inspection policies

3.1 Block replacement and block inspection

Our base policy is as follows: preventively replace all components at times lT , $l=1,2,\dots$; schedule inspections for all components to occur at times $lT+j\Delta$, $j=1,\dots,K$, ($K\Delta < T$); at inspection, if component i is defective, then replace it; on failure of component i , replace it. Thus, this policy combines block replacements and block inspections, and the corrective maintenance that is carried out when defectives are found at inspection or failures occur is, at most, that which is required to return all components to the good state. Label this policy BB, denoting block replacement and block inspections. The policy thus comprises a series of preventive replacement cycles within which there is an initial inspection phase of length $K\Delta$ and subsequent non-inspection or wear out phase of length $T - K\Delta$. Note that, when a defective or failed component is replaced a renewal of that component occurs, but the system itself is not renewed (for the cases where the number of components > 1).

We now consider variations on this policy which have an opportunistic character. That is, a replacement of a failed component is considered as an opportunity to inspect all others. As an aside, provided that inspections are effective (in determining whether or not the component is in the good state), a policy which considers a replacement of a failed component as an opportunity to replace all other components could not reasonably be justified in practice due to the likely high cost of replacement relative to the cost of inspection. On the other hand, we might let the probability that an inspection is ineffective (does not reveal the defective state) be p , and consider for what values of p such a policy is economically justified. However returning to the perfect inspection case, consider the policy: as the base policy, plus on failure of component i , inspect all other components. Label this policy BB+.

We might vary this policy in order to carry out the opportunistic inspections only if the system is in the inspection phase. Thus label as BBI+ the following policy: as the base policy, plus on failure of component i at time $lT + z_i$, inspect all other components if $z_i < K\Delta$ (any l).

3.2 Block replacement and age based inspection

Next we consider policies in which the replacements are block type (that is, periodic), but the inspections are related to the age of the component or system (that is, the time since last replacement). These policies will be similar to the base policy, except that on replacement of component i the inspections for that component are rescheduled from the time of the replacement. Thus, inspections of components in the system will then go out of phase—this will be more costly, but may lead to fewer failures (of operational function) on average. Thus consider the policy: as the base policy, plus on replacement of component i at time $lT + z_i$ (either due to it being defective at inspection or due to failure), reschedule at most K further inspections for component i , that is inspect component i at $lT + z_i + j\Delta$, $j = 1, \dots, \min(K, K')$ where $K' = \sup(j : z_i + j\Delta < T)$. Label this policy BA1.

We might also consider age based replacement and inspection for this multi-component system. However, replacements will soon go out of synchronization, and consequently, due to the specification of the cost structure in which there is a shared set-up cost for all inspections and replacements, such an age-age policy will become prohibitively costly. When the set-up is small, then this policy may have some merit. When the set-up cost is zero, there are no interactions between components, in the sense of Dekker et al. [8], and we need not consider the system as a multi-component system but as n independent one-component systems.

Failure interactions, in which for example a defect in one component may influence the failure process in another component [9,10], are not considered, although the consideration of hybrid inspection and replacement policies in this context might make an interesting study.

A variation on the BA1 policy would reschedule inspections of other components to coincide with the new inspection schedule of the replaced component. That is, as the base policy plus on replacement of component i at time $lT + z_i$, reschedule inspections of all components to occur at $lT + z_i + j\Delta$, $j = 1, \dots, \min(K, K')$ where $K' = \sup(j : z_i + j\Delta < T)$. Label this policy BAn. If the replacement at time $lT + z_i$ is due to failure, then we might also inspect all other components at this time. Label this policy BAn+. The policy BA1+ is another variation.

A further variation would reschedule the inspections only if the system was in the inspection phase. That is, as the base policy plus, on replacement of component i at time $lT + z_i$, if $z_i < K\Delta$ reschedule inspections of all components to occur at $lT + z_i + j\Delta$, $j = 1, \dots, \min(K, K')$ where $K' = \sup(j : z_i + j\Delta < T)$. Label this policy BAI n , and the similar policy that inspects other components at the time of failure of component i as BAI n +

3.2 Block replacement and age based inspection with in-phase rescheduling

Alternatively, on failure of component i at time $lT + z_i$, we might reschedule inspections of component i or all components to occur at times $lT + j\Delta$, $j = j', \dots, j' + j'' - 1$ where $j' = \inf(j : j\Delta > z_i)$ and $j'' = \min\{K - 1, \sup(j : z_i + j\Delta < T)\}$ provided $T - z_i > \Delta$, rather than at

times $lT + z_i + j\Delta$, $j = 1, \dots, \min(K, K')$ —that is, to occur in phase rather out of phase with the original inspection schedule (policies BAR1, BAR n). The rescheduling might also be carried out only if the failure occurs during the inspection phase (BAR1 l , BAR ln). We might also inspect all other components at the time of failure of component i (BAR1 $l+$, BAR $ln+$).

There are a large number of policies here and engineering judgement about the particular context will narrow the number of policies to investigate. Some would be considered impractical. Others would have cost advantages that were immediately apparent. We therefore do not analyse the cost and reliability characteristics of all policies, but concentrate on a small subset.

3.4 Modified block replacement and age based inspection with in-phase rescheduling

In the above policies, in practice one would expect the decision about inspecting all other components on failure of a single component to depend on whether the failure occurs early in the inspection phase or not. A failure early in the inspection phase might be regarded as a good opportunity to inspect other components; a failure late in the inspection phase might be considered a waste of effort. Conversely, a failure late in the wear-out phase might be a good opportunity to inspect other components. These types of adaptations then suggest considering additional policies that have parallels with the modified block replacement policy [11]. These types of policies would be simpler to implement than the block replacement, age based inspection hybrids described in sections 3.2 and 3.3, and would be cost efficient with respect to the block-block hybrid policies (BB) in section 3.1.

4. Analysis of the hybrid policy BB

For the standard block replacement policy (replace on failure and every T time units) for an n component series system, the long run cost per unit time is given by $\{(c_S + c_R + c_F)nM(T) + c_S + nc_R\}/T$ where $M(T) = \sum_{i=1}^{\infty} F_Y^{(i)}(T)$ is the expected number of failures in $[0, T]$ and $F_Y^{(i)}(t)$ is the i -fold convolution of $F_Y(t)$ with itself ([12], p.119). The renewal theory argument used to obtain this result will be very difficult to adapt even to the most straightforward of the policies that we consider. This is because we would need to determine the expected number of defects and the expected number of failures in each inspection interval, $[(i-1)\Delta, i\Delta]$, $i = 1, \dots, K$, and over the wear out phase, $(K\Delta, T]$, and furthermore defectives and failures are not independent. Therefore, we attempt to approximate the long run cost per unit time by allowing only a finite and small number of defectives and failures during a preventive replacement cycle. This approximation will be reasonable only if K, Δ , and T are small, so that defectives and failures are rare.

Denote the number of defectives found at inspection and the number of failures in a single component during a preventive replacement cycle $[0, T]$ by N_D and N_F , respectively. For the BB policy, it follows that

$$\begin{aligned}
\text{prob}(N_D = N_F = 0) &= \text{prob}(X > K\Delta, X + Y > T) \\
&= \text{prob}\{(K\Delta < X < T \cap Y \geq T - X) \cup (X > T)\} \\
&= \int_{K\Delta}^T R_Y(T - x) dF_X + R_X(T) \\
&= p_{00},
\end{aligned}$$

say, where $R(x) = 1 - F(x)$ is the reliability function (and X is the component age at defect arrival, and Y is the corresponding delay time from defect arrival to failure). Further,

$$\begin{aligned}
\text{prob}(N_D = 1, N_F = 0) &= \sum_{i=1}^K \text{prob}\{(i-1)\Delta < X_1 < i\Delta, X_1 + Y_1 > i\Delta, X_2 > (K-i)\Delta, X_2 + Y_2 > T - i\Delta\} \\
&= \sum_{i=1}^K \int_{(i-1)\Delta}^{i\Delta} R_{Y_1}(i\Delta - x_1) dF_{X_1} \left\{ \int_{(K-i)\Delta}^{T-i\Delta} R_{Y_2}(T - i\Delta - x_2) dF_{X_2} + R_{X_2}(T - i\Delta) \right\} \\
&= p_{10}.
\end{aligned}$$

($K > 0$). When $K=0$, $p_{10}=0$, as a defect can be found only if there is at least 1 inspection. (For the i th defect, X_i is the age at arrival and Y_i is the delay time, $i=1,2$.) The probability terms in the summation in the right hand side of the above equation correspond to the cases in which the defect arises in the first inspection interval, in the second, ..., in the K th (if the defect arises in the first, then there must be no arrival from the second inspection onward,...). Further,

$$\begin{aligned}
\text{prob}(N_D = 0, N_F = 1) &= \sum_{i=1}^K \text{prob}\{(i-1)\Delta < X_1 < i\Delta, X_1 + Y_1 < i\Delta, X_1 + Y_1 + X_2 > K\Delta, X_1 + Y_1 + X_2 + Y_2 > T\} \\
&\quad + \text{prob}\{K\Delta < X_1 < T, X_1 + Y_1 < T, X_1 + Y_1 + X_2 + Y_2 > T\} \\
&= \sum_{i=1}^K \int_{(i-1)\Delta}^{i\Delta} \int_0^{i\Delta-x_1} \int_{K\Delta-x_1-y_1}^{T-x_1-y_1} \{ \int_{K\Delta-x_1-y_1}^{T-x_1-y_1} R_{Y_2}(T - x_1 - y_1 - x_2) dF_{X_2} + R_{X_2}(T - x_1 - y_1) \} dF_{Y_1} dF_{X_1} \\
&\quad + \int_{K\Delta}^T \int_0^{T-x_1} \int_0^{T-x_1-y_1} \{ \int_0^{T-x_1-y_1} R_{Y_2}(T - x_1 - y_1 - x_2) dF_{X_2} + R_{X_2}(T - x_1 - y_1) \} dF_{Y_1} dF_{X_1} \\
&= p_{01},
\end{aligned}$$

(for $K > 0$). When $K=0$, the first term on the above expression is 0.

We could add a $N_D=1, N_F=1$ term, and call this p_{11} , although we might rule out the possibility that the defect and the failure arise in the same inspection interval, or we might suppose that the failure can only arise in $(K\Delta, T]$ —this will make the calculation simpler. In principal, the probability calculation would be similar to above.

Further, to calculate $\text{prob}(N_D = 0, N_F = 2)$, we need terms for: 1 failure in $[0, K\Delta]$ and 1 failure in $(K\Delta, T]$; 2 failures in $[0, K\Delta]$ and 0 failure in $(K\Delta, T]$; 0 failure in $[0, K\Delta]$ and 2 failures on $(K\Delta, T]$. Denoting these by ${}_1p_{02}$, ${}_2p_{02}$, and ${}_3p_{02}$ respectively, we have that

$$\text{prob}(N_D = 0, N_F = 2) = {}_1p_{02} + {}_2p_{02} + {}_3p_{02} = p_{02} \quad (1)$$

where

$$\begin{aligned}
{}_1 p_{02} &= \sum_{i=1}^K \text{prob}\{(i-1)\Delta < X_1 < i\Delta, X_1 + Y_1 < i\Delta, K\Delta < X_1 + Y_1 + X_2, X_1 + Y_1 + X_2 + Y_2 < T, \\
&\quad X_1 + Y_1 + X_2 + Y_2 + X_3 + Y_3 > T\} \\
&= \sum_{i=1}^K \int_{(i-1)\Delta}^{i\Delta} \int_0^{i\Delta-x_1} \int_{K\Delta-x_1-y_1}^{T-x_1-y_1} \int_0^{T-x_1-y_1-x_2} \left\{ \int_0^{T-x_1-y_1-x_2-y_2} R_{Y_3}(T-x_1-y_1-x_2-y_2-x_3) dF_{X_3} \right. \\
&\quad \left. + R_{X_3}(T-x_1-y_1-x_2-y_2) \right\} dF_{Y_2} dF_{X_2} dF_{Y_1} dF_{X_1},
\end{aligned}$$

for $K>0$. If $K=0$, then ${}_1 p_{02}=0$.

$$\begin{aligned}
{}_2 p_{02} &= \sum_{i=1}^K \sum_{j=i}^K \text{prob}\{(i-1)\Delta < X_1 < i\Delta, X_1 + Y_1 < i\Delta, \\
&\quad (j-1)\Delta < X_1 + Y_1 + X_2 < j\Delta, X_1 + Y_1 + X_2 + Y_2 < j\Delta, \\
&\quad X_1 + Y_1 + X_2 + Y_2 + X_3 > K\Delta, X_1 + Y_1 + X_2 + Y_2 + X_3 + Y_3 > T\} \\
&= \sum_{i=1}^K \sum_{j=i}^K \int_{(i-1)\Delta}^{i\Delta} \int_0^{i\Delta-x_1} \int_{\max\{0, (j-1)\Delta-x_1-y_1\}}^{j\Delta-x_1-y_1} \int_0^{j\Delta-x_1-y_1-x_2} \left\{ \int_{K\Delta-x_1-y_1-x_2-y_2}^{T-x_1-y_1-x_2-y_2} R_{Y_3}(T-x_1-y_1-x_2-y_2-x_3) dF_{X_3} \right. \\
&\quad \left. + R_{X_3}(T-x_1-y_1-x_2-y_2) \right\} dF_{Y_2} dF_{X_2} dF_{Y_1} dF_{X_1},
\end{aligned}$$

for $K>0$. If $K=0$, then ${}_2 p_{02}=0$. The quantity $\max\{0, (j-1)\Delta-x_1-y_1\}$ in the lower limit of the third integral handles the case in which the two failures arise in the same inspection interval, whence the value of $\max\{0, (j-1)\Delta-x_1-y_1\}$ is 0.

$$\begin{aligned}
{}_3 p_{02} &= \text{prob}\{K\Delta < X_1 < T, X_1 + Y_1 < T, X_1 + Y_1 + X_2 + Y_2 < T, X_1 + Y_1 + X_2 + Y_2 + X_3 + Y_3 > T\} \\
&= \int_{K\Delta}^T \int_0^{T-x_1} \int_0^{T-x_1-y_1} \int_0^{T-x_1-y_1-x_2} \left\{ \int_0^{T-x_1-y_1-x_2-y_2} R_{Y_3}(T-x_1-y_1-x_2-y_2-x_3) dF_{X_3} \right. \\
&\quad \left. + R_{X_3}(T-x_1-y_1-x_2-y_2) \right\} dF_{Y_2} dF_{X_2} dF_{Y_1} dF_{X_1}.
\end{aligned}$$

(for $K \geq 0$).

Thus $\text{prob}(N_D = 0, N_F = 2)$ is quite a large expression, and rather slow to enumerate. In developmental work, we found that the approximation using only p_{00} , p_{01} and p_{10} was not particularly good (based on a comparison with simulated results). We therefore include the p_{02} term, and also further the p_{11} term. To calculate p_{11} , we consider 3 terms again. In the first, the defect and failure occur within the inspection period and the defect occurs before the failure. This occurs with probability

$$\begin{aligned}
{}_1 p_{11} &= \sum_{i=1}^{K-1} \int_{(i-1)\Delta}^{i\Delta} R_{Y_1}(i\Delta-x_1) dF_{X_1} \left\{ \sum_{j=i+1}^K \int_{(j-i-1)\Delta}^{(j-i)\Delta} \int_0^{(j-i)\Delta-x_2} \left\{ \int_{(K-i)\Delta-x_2-y_2}^{T-i\Delta-x_2-y_2} R_{Y_3}(T-i\Delta-x_2-y_2-x_3) dF_{X_3} \right. \right. \\
&\quad \left. \left. + R_{X_3}(T-i\Delta-x_2-y_2) \right\} dF_{Y_2} dF_{X_2} \right\},
\end{aligned}$$

($K>1$). If $K=1$, then ${}_1 p_{11}=0$. In the second term, the failure occurs before the defect with probability

$$\begin{aligned}
{}_2 p_{11} &= \sum_{i=1}^K \sum_{j=i}^K \int_{(i-1)\Delta}^{i\Delta} \int_0^{i\Delta-x_1} \int_{\max[0, (j-i)\Delta-x_1-y_1]}^{j\Delta-x_1-y_1} R_{Y_2}(j\Delta-x_1-y_1-x_2) dF_{X_2} dF_{Y_1} dF_{X_1} \times \\
&\quad \left\{ \int_{(K-j)\Delta}^{T-j\Delta} R_{Y_3}(T-j\Delta-x_3) dF_{X_3} + R_{X_3}(T-j\Delta) \right\}.
\end{aligned}$$

($K>0$). In the third, the failure occurs after the K th inspection with probability:

$${}_3p_{11} = \sum_{i=1}^K \int_{(i-1)\Delta}^{i\Delta} R_{Y_1}(i\Delta - x_1) dF_{X_1} \left\{ \int_{(K-i)\Delta}^{T-i\Delta} \int_0^{T-i\Delta-x_2} \left\{ \int_0^{T-i\Delta-x_2-y_2} R_{Y_3}(T-i\Delta-x_2-y_2-x_3) dF_{X_3} \right. \right. \\ \left. \left. + R_{X_3}(T-i\Delta-x_2-y_2) \right\} dF_{Y_2} dF_{X_2} \right\}.$$

($K>0$), and so

$$\text{prob}(N_D = 1, N_F = 1) = {}_1p_{11} + {}_2p_{11} + {}_3p_{11} = p_{11}$$

If $K=0$, then $p_{11}=0$, as a defect can be found only if there is at least 1 inspection.

The long run cost per unit time for the block inspection, block replacement hybrid (BB) policy can now be approximated as follows. Suppose a single component over a cycle can only be subject to: 0 defects (found at inspection) and 0 failures (with probability p_{00}); or 0 defects and 1 failure (with probability p_{01}); or 1 defect and 0 failures (with probability p_{10}); or 1 defect and 1 failure (with probability p_{11}); or 0 defect and 2 failures (with probability p_{02}). In any cycle, let the (random) number of components in each category among the n components in the system be n_{00} , n_{01} , n_{10} , n_{11} , and n_{20} respectively. Then, the cost of the event $(n_{00}, n_{01}, n_{10}, n_{11}, n_{02})$ will be

$$C(n, n_{01}, n_{10}, n_{11}, n_{02}) = K(nc_1 + c_S) + c_S + nc_R + (n_{01} + n_{11} + 2n_{02})(c_S + c_R + c_F) + (n_{10} + n_{11})c_R.$$

The terms in this cost arise as a result of: K inspections for all n components; preventive replacement of all n components at the cycle end (block replacement); $(n_{01} + n_{11} + 2n_{02})$ failures in total, each with cost $(c_S + c_R + c_F)$; and $(n_{10} + n_{11})$ defects found at inspection, each with cost c_R . In the final term, there is no additional shared set-up cost because, at the time a defect is found, all components are undergoing inspection. The cost above will be incurred with probability

$$P(n, n_{01}, n_{10}, n_{11}, n_{02}) = \frac{n!}{(n - n_{01} - n_{10} - n_{11} - n_{02})! n_{01}! n_{10}! n_{11}! n_{02}!} p_{00}^{(n - n_{01} - n_{10} - n_{11} - n_{02})} p_{01}^{n_{01}} p_{10}^{n_{10}} p_{11}^{n_{11}} p_{02}^{n_{02}}$$

(the multinomial probability of the event $(n_{00}, n_{01}, n_{10}, n_{11}, n_{02})$). The long run (average) cost per unit time is then just

$$C_\infty(T) = \left\{ \sum_{n_{01}=0}^n \sum_{n_{10}=0}^{n-n_{01}} \sum_{n_{11}=0}^{n-n_{01}-n_{10}} \sum_{n_{02}=0}^{n-n_{01}-n_{10}-n_{11}} C(n, n_{01}, n_{10}, n_{11}, n_{02}) P(n, n_{01}, n_{10}, n_{11}, n_{02}) \right\} / T. \quad (2)$$

This expression will be straightforward to calculate provided n is not too large, although computationally quite slow. In fact, in the application that we consider later in the paper, we found that it was impractical to calculate the term p_{02} using the expression (1) above. Instead we set

$$p_{02} = 1 - p_{00} - p_{10} - p_{01} - p_{11}, \quad (3)$$

having first calculated p_{00} , p_{01} , p_{10} , and p_{11} . In so doing, we are, in effect, replacing p_{02} with $p_{02} + p_{12} + p_{22} + \dots$, and so we will obtain a better approximation than if we use the value for p_{02} in expression (1). Since $c_{02} < c_{12}, c_{22}, \dots$ (where c_{02}, c_{12}, \dots refer to the cost over a cycle of a single socket in which there are 2 failures, 1 defect and 2 failures, etc.), we will still be underestimating the total cost slightly.

The mean time between operational failures can be obtained by approximation whence

$$\mu = T / n(p_{01} + p_{11} + 2p_{02}) \quad (4)$$

since under our simplifying assumptions (at most 2 failures per component in the interval $[0, T]$), the expected number of failures in $[0, T]$ is $n(p_{01} + p_{11} + 2p_{02})$.

5. Analysis of the more complex hybrid policies

The list of policies that could be investigated is rather large. Some of these policies have practical implications for the management of maintenance planning. The block replacement, age based replacement hybrids would be difficult manage. An inspection schedule for the system would have to be updated if a defect is found at inspection or if a failure occurs. In the latter case, the emphasis of maintainers may be to return the system as quickly as possible to the operational state. Additional inspections will then be undesirable, and so the policies that inspect all components on failure may be uninteresting. This then rules out all the (+) policies using the notation of section 3, unless c_1 is small relative to c_S .

The block, age hybrid policies for which rescheduled inspections are not brought into phase with the existing schedule (BA1, BAn, BAI1, BAI n) will be very difficult to implement in practice. Therefore it makes sense to focus on the block, age hybrids for which the rescheduled inspections are brought in phase with the inspections of other components (BAR1, BAR n , BARI1, BARI n). Further practical considerations might mean that rescheduling the inspection of all components on failure of one may be sensible, particularly if the set-up cost is large relative to the inspection cost (for a single component. This would then rule out the -1 policies from further investigation, unless c_1 is large relative to c_S .

The block, block hybrid policy BB is the simplest to optimize. In fact, analytical calculation of the long run cost and the mean time between operational failures is intractable for all but the BB policy. Also, simulation of these policies is also difficult. The other policies will no doubt be cost-efficient with respect to the BB policy. The question is, how much more efficient? Some broad, heuristic argument about the number of failures that might be prevented and the number of extra inspections if, say, the BAR n policy is used in place of BB.

For example, for the BB policy, on average there are (approximately) $T/\mu(\text{BB})$ failures in a replacement cycle of length T , and where $\mu(\text{BB})$ is the mean time between operational failures associated with this policy. Now, on failure, if the new component is weak, under the BB policy this component has a high chance of failure (say, $\frac{1}{2}$), since its lifetime is likely to be less than residual time to end of the block replacement interval. Thus the BAR n policy will prevent an additional $\frac{1}{2} pT/\mu(\text{BB})$ failures, at cost c_F each. BAR n will incur approximately $\frac{1}{2} K \times (T/\mu(\text{BB}))$ additional inspections, which will cost $(nc_1 + c_S)$ each. Thus $C(\text{BAR}n) < C(\text{BB})$ (in an obvious notation) if

$$c_F p > K(nc_1 + c_S) \quad (5)$$

We might like to compare BAR1 with BAR n —that is only rescheduling the inspection of the new component (following failure) and not all components. When operating the BAR n policy as opposed to the BAR1 policy, we would expect to prevent

$$\frac{T}{\mu(\text{BB}, K^*)} - \frac{T}{\mu(\text{BB}, \frac{3}{2} K^*)}$$

additional failures, where $\mu(\text{BB}, K)$ is the MTBOF of the BB policy when there are K inspections (and other decision variables are held at their optimum values). This is because, broadly, at each failure the $\text{BAR}n$ policy extends the inspection phase from $[0, K\Delta]$ to $[0, \frac{3}{2} K\Delta]$, relative to the $\text{BAR}1$ policy. Now the $\text{BAR}n$ policy incurs approximately an additional $\frac{1}{2} K \times (n-1)$ inspections for every failure. There will be approximately $T / \mu(\text{BB})$ failures in a replacement interval, and so $C(\text{BAR}n) > C(\text{BAR}1)$ if

$$\frac{1}{2} K (n-1) c_I \frac{T}{\mu(\text{BB}, K^*)} > \left\{ \frac{T}{\mu(\text{BB}, K^*)} - \frac{T}{\mu(\text{BB}, \frac{3}{2} K^*)} \right\} c_F.$$

That is, if

$$c_I > \frac{2c_F}{K(n-1)} \left\{ 1 - \frac{\mu(\text{BB}, K^*)}{\mu(\text{BB}, \frac{3}{2} K^*)} \right\}, \quad (6)$$

which is written in terms of the cost of inspection since it is this that we would expect to drive the choice between $\text{BAR}n$ and $\text{BAR}1$.

The $\text{BAR}In$ policy reschedules inspections only if failure occurs during the inspection phase. In a sense this policy is a compromise between BB and $\text{BAR}n$ and so its cost and its MTBOF will lie between that of BB and $\text{BAR}n$. This then allows us to get a handle on this policy.

So far we have only discussed the total cost and the MTBOF of the more complex policies. For the optimum values of decision variables, it is likely that these will be very similar to the BB policy. This is because the long-run cost function for the policies will be generally very flat in region of optimum [13].

To summarise, for calculation purposes, we focus on the BB policy, and use both the numerical approximation of section 4 and simulation. We do this in the context of a real multi-component system in the following section. Consideration of this case allows us to determine realistic values for the cost and reliability criteria. Given the complexity of the $\text{BAR}1$, $\text{BAR}n$ and $\text{BAR}In$ policies, we only consider these in the approximate, heuristic sense discussed above.

6. Numerical study

In order to illustrate the policies of interest (BB, $\text{BAR}n$, $\text{BAR}1$), we consider the failure of 375V d.c. traction motor bearings on a commuter railway. Data collected on the lifetimes of bearings (figure 1) were modelled by Scarf et al. [2] using a mixture of Weibull distributions for the time to defect arrival, with parameters $p=0.10$, $\eta_1=2.5$ (years), $\beta_1=3$, $\eta_2=18$, $\beta_2=5$. An exponential distribution is arbitrarily chosen for the delay times, with mean $\lambda=1/2$ (year), implying a characteristic life of approximately 3 years for weak components and 18.5 years for strong components. The maintenance policy of the railway was to replace motors after 5 to 7 years, and therefore we would expect to see only very few failures of strong (long-lived) components.

At the time of the data collection, the company used of the order of 2300 motors. We simplify the context somewhat in order make the numerical calculations manageable, and suppose that an individual train is an 8-component-series system. Interactions between bearings and other

components of the train are ignored. For a fuller discussion of the system see [14]. Costs are considered relative to the cost of a preventive replacement. Thus we take $c_R=1$. Arbitrarily we take the cost of failure $c_F=10$. In [2] the cost of inspection for a single component system was taken to be 0.05 in the base case. For consistency, we take the set-up cost to be $c_S=0.2$, and the inspection cost $c_I=0.025$. These values imply, for an 8-component system, a total inspection cost of $0.2+8*0.025=0.4$, and hence a per component cost of 0.05 units.

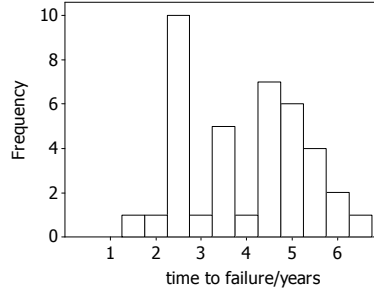


Figure 1. Histogram of ages at bearing failure for 39 traction motors among the population of 2296.

6.1. Results for the BB policy

For the BB policy, we use simulation and the numerical approximation developed in section 4 to calculate the long run cost per unit time and the mean time between operational failures (MTBOF). We begin by comparing the simulation and the numerical approximation. Figures 2 and 3 provide a reasonable validation of both approaches. They also show how the approximation performs as it is refined with the inclusion of extra terms. For the base parameter set, using $p_{02} = 1 - p_{00} - p_{10} - p_{01} - p_{11}$ gave $p_{02} = 7.036 \times 10^{-3}$ at the optimum values of the decision variables. Using the expression (1) we obtain $p_{02} = 5.867 \times 10^{-3}$. The difference is 0.0011, and so we are underestimating the total cost in a typical cycle by approximately $c_F \times 0.0011$. Thus, with a typical cycle length $T=10$, and $c_F=10$, the long-run cost per unit time is underestimated by approximately 0.001, which is less than a 0.1% error—c.f. with the values in table 1.

It should be noted that the simulation itself provides only an approximation. This is because we have to impose finite limits on the numbers of possible defects and failures during a cycle. If these limits are large then the simulation will be less biased and hence more accurate. The bias arises because the sum of the probabilities of the possible (mutually exclusive) outcomes in a cycle are slightly less than one. In the numerical approximation we correct for this.

In figures 2 and 3, we also illustrate the progressive improvement in the approximation to the long run cost, and the MTBOF, as additional terms are added. The simplest approximation that we considered in exploratory work calculated p_{00} , p_{10} , and p_{01} and then set

$$p_{11} = 1 - p_{00} - p_{10} - p_{01}. \quad (7)$$

The long run cost per unit time and MTBOF were then calculated using

$$P(n, n_{01}, n_{10}, n_{11}) = \frac{n!}{(n - n_{01} - n_{10} - n_{11})! n_{01}! n_{10}! n_{11}!} p_{00}^{(n - n_{01} - n_{10} - n_{11})} p_{01}^{n_{01}} p_{10}^{n_{10}} p_{11}^{n_{11}},$$

$$C_{\infty}(T) = \{ \sum_{n_{01}=0}^n \sum_{n_{10}=0}^{n-n_{01}} \sum_{n_{11}=0}^{n-n_{01}-n_{10}} C(n, n_{01}, n_{10}, n_{11}) P(n, n_{01}, n_{10}, n_{11}) \} / T, \quad (8)$$

and

$$\mu = T / n(p_{01} + p_{11}). \quad (9)$$

This approximation is shown as the bold line. While the cost per unit time is significantly underestimated, the implied “optimum” values of the decision variables appear to be little changed. Thus one would expect that even this approximation would be reasonable when choosing the optimal policy.

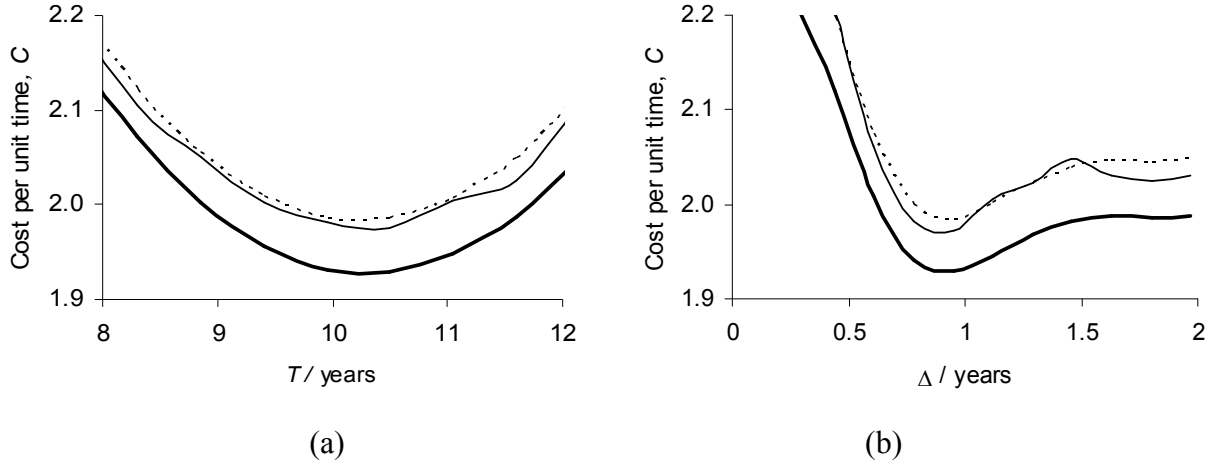


Figure 2. For hybrid BB policy, long run cost per unit time, $C(T)$: (a) as a function of block replacement interval T with $\Delta=0.85$, $K=5$ (minimum cost values); (b) as a function of inspection interval Δ with $T=10.24$, $K=5$ (minimum cost values). $C(T)$ obtained using simulation (—), by the approximation, equations (2) and (3) (-----), and by the simpler approximation, equations (7) and (8) (—). Parameter values: $p=0.1$, $\eta_1=3$, $\eta_2=18$, $\beta_1=2.5$, $\beta_2=5$, $\lambda=1/2$, $c_R=1$, $c_F=10$, $c_I=0.025$, $c_S=0.2$.

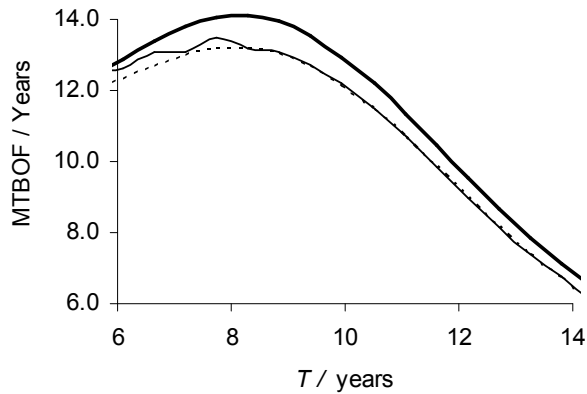


Figure 3. For hybrid BB policy, mean time between operational failures (MTBOF) as a function of block replacement interval T with $\Delta=0.85$, $K=5$ (minimum cost values). $C(T)$ obtained using simulation (—), by the approximation, equations (2) and (3) (-----), and by the simpler approximation, equations (7) and (9) (—). Parameter values: $p=0.1$, $\eta_1=3$, $\eta_2=18$, $\beta_1=2.5$, $\beta_2=5$, $\lambda=1/2$.

Figures 4 and 5 show how the cost and system reliability (MTBOF) of the hybrid BB policy varies with values of the decision variables in the base case (parameter values as indicated). The refined approximation is used here. We can observe that the decision criteria are quite sensitive to the value of Δ and more so than to T . Thus, for a given K (number of inspections), Δ needs to be chosen fairly carefully.

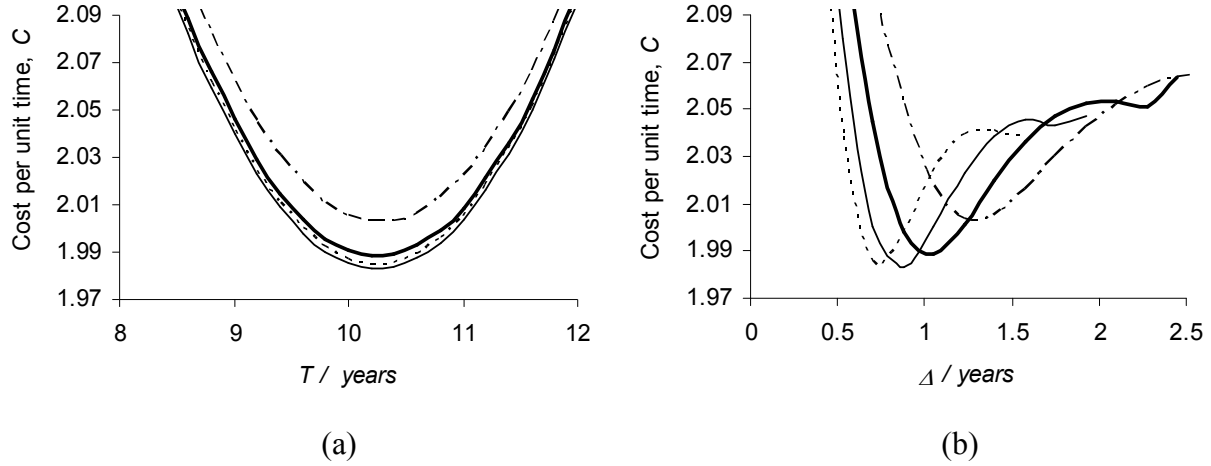


Figure 4. For hybrid BB policy, long run cost per unit time, $C_\infty(T, \Delta, K)$ calculated using the approximation, equations (2) and (3): (a) as a function of the block replacement interval, T ; (b) as a function of inspection interval, Δ ; for $K=3$ (---), $K=4$ (—), $K=5$ (—), $K=6$ (-----). Parameter values: $\eta_1=3$, $\eta_2=18$, $\beta_1=2.5$, $\beta_2=5$, $p=0.1$, $\lambda=1/2$, $c_R=1$, $c_F=10$, $c_I=0.025$, $c_S=0.2$. Non-varying decision variable held at minimum cost value.

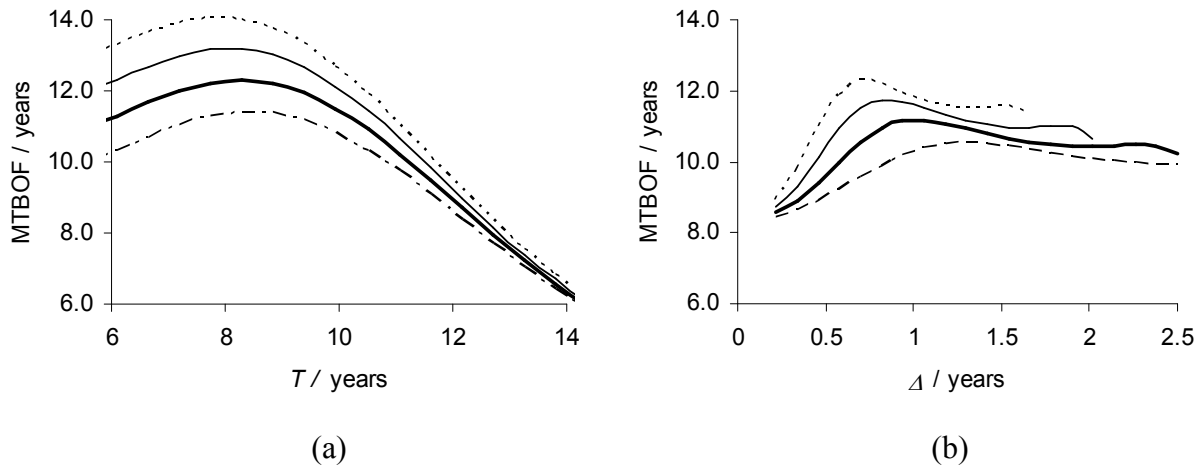


Figure 5. For hybrid BB policy, mean time between operational failures (MTBOF): (a) against T ; (b) against Δ ; using the approximation, equations (3) and (4). For $K=3$ (---), $K=4$ (—), $K=5$ (—), $K=6$ (-----). Parameter values: $\eta_1=3$, $\eta_2=18$, $\beta_1=2.5$, $\beta_2=5$, $p=0.1$, $\lambda=1/2$. Non-varying decision variable held at minimum cost value.

In table 1, we vary the parameter values. All calculations are carried out using the numerical approximation, equations (2) and (3). The first row presents the base case. Variations on the base case are highlighted. The policy appears to behave as expected. For example, the inspection frequency increases as the cost of inspection decreases and as the set up cost decreases. The block replacement interval, T , is sensitive to the cost of failure and to the characteristic lifetime, η_2 , of components arising from the strong sub-population. Again this is as we would expect. The optimum policy appears to be quite sensitive to the value of the mixing parameter, p . On the other hand, the optimum policy is relatively insensitive to the separation of the sub-populations in the mixture. This is in contrast to the equivalent hybrid age based policy [2]—this latter policy was quite sensitive to the separation of the sub-populations, and for even moderately larger values of η_1 above the base case, it was optimal to inspect over the entire replacement interval $[0, T]$.

Table 1. Optimum hybrid BB policy for various values of cost parameters and failure model parameters. Long run cost per unit time for the optimum policy is C^* . Unit cost equal to the cost of preventive replacement, c_R . Time unit here 1 year.

β_1	η_1	β_2	η_2	p	λ	c_I	c_S	c_R	c_F	C^*	T^*	Δ^*	K^*
2.5	3	5	18	0.1	0.5	0.025	0.2	1	10	1.982	10.24	0.85	5
2	3	5	18	0.1	0.5	0.025	0.2	1	10	1.999	10.24	0.87	5
3	3	5	18	0.1	0.5	0.025	0.2	1	10	1.970	10.24	0.82	5
2.5	2	5	18	0.1	0.5	0.025	0.2	1	10	1.914	10.11	0.59	5
2.5	4	5	18	0.1	0.5	0.025	0.2	1	10	2.017	10.24	1.13	5
2.5	3	4	18	0.1	0.5	0.025	0.2	1	10	2.264	9.44	0.88	5
2.5	3	6	18	0.1	0.5	0.025	0.2	1	10	1.797	10.91	0.84	5
2.5	3	5	12	0.1	0.5	0.025	0.2	1	10	2.782	7.25	0.83	8
2.5	3	5	24	0.1	0.5	0.025	0.2	1	10	1.512	13.53	0.84	5
2.5	3	5	18	0.1	0.25	0.025	0.2	1	10	1.161	10.25	2.73	1
2.5	3	5	18	0.1	1	0.025	0.2	1	10	1.770	10.35	0.94	5
2.5	3	5	18	0.05	0.5	0.025	0.2	1	10	1.618	9.67	2.95	1
2.5	3	5	18	0.15	0.5	0.025	0.2	1	10	2.312	10.71	0.64	7
2.5	3	5	18	0.1	0.5	0.025	0.1	1	10	1.906	10.16	0.64	7
2.5	3	5	18	0.1	0.5	0.025	0.3	1	10	1.983	10.23	0.85	3
2.5	3	5	18	0.1	0.5	0.025	0.2	1	2	1.124	12.29		0
2.5	3	5	18	0.1	0.5	0.025	0.2	1	20	2.714	9.41	0.52	9

6.2. BAR1 and BARn policies

Using the heuristic arguments of section 5, for the base parameter set, using equation (5) we would broadly expect the BARn policy to be more cost efficient if $c_F > 20$. The BAR1 policy would be still more efficient if $c_I > 0.1$ (from equation 6). Thus if the cost of failure is high then it makes sense to do additional inspections subsequent to a failure. Further, if the cost of inspection is high then one would only do additional inspections of the new component. For the cost of failure we use ($c_F = 10$), BARn would not be optimal. Then the question arises: is BAR1 more efficient than the BB policy? Broadly speaking, BAR1 will always be more cost-efficient than BARn and BB.

However, it will also be more difficult to manage than the BAR_n policy which itself will be more difficult to manage than the BB policy. The BAR_n policy, which reschedules inspection on failure or when a defect is found but only if this occurs during the inspection phase, is likely to be cost efficient in comparison to BB, BAR_n , and BAR_1 for moderate values of the cost of failure ($10 < c_F < 20$) and provided the cost of inspection was not unusually large ($c_I < 0.1$).

7. Discussion

In this paper, for an n -component series system in which the component parts used at successive replacements arise from a heterogeneous population, we present novel replacement policies that are hybrids of inspection maintenance and block replacement. Heterogeneity is modelled using a mixed distribution for the time to failure of a component. A three state failure model allows us to consider inspection maintenance. A hybrid of the pure block replacement policy and pure block inspection policy is the simplest. The behaviour of this policy is illustrated using data relating to the maintenance of traction motor bearings on underground trains. We also compare variations on the simple policy, and show that for moderate cost of failure and cost of inspection the simple policy (BB policy) is the most sensible policy to use in practice. Among the more complex policies, which are opportunistic in nature, we would expect to find a policy that demonstrates a cost saving and an increase in the mean time between operational failures in comparison to the simplest policy. However, the opportunistic policy is more complex and will be more difficult to manage operationally.

It is interesting to note that the system has the property that there is an upper bound on its reliability as measured by the mean (or median) time between operational failures. This is because early preventive maintenance may be counter productive since one could introduce a weak item in place of a good item. Thus, the model developed in a sense can explain the effect of poor maintenance. The model also captures the notion that newer equipment tends to be looked after more carefully than older equipment.

The calculation of long run cost and system reliability is difficult for all but the simplest policy. Even for this policy, we use an approximation that assumes only a limited number defects and failures can arise in a component. On comparison with a simulation of the system, the simplifying approximation appears to perform quite well.

Numerous policies of the type we describe might be investigated. We confine ourselves here to a small number of policies that would be practical to implement. We consider only the economic interaction between components in the system—that is interactions that influence the cost structure. It may be of interest to extend the policies to cases in which there is failure interaction, although this would be of most interest in the context of a real case study in which such interactions arise.

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References

- [1] Jiang R, Jardine AKS. An optimal burn-in preventive-replacement model associated with a mixture distribution. *Quality and Reliability Engineering International* 2007; 23:83–93.
- [2] Scarf PA, Cavalcante CAV, Dwight R, Gordon P. An age-based inspection and replacement policy for heterogeneous components. *IEEE Transactions on Reliability*, 2009, to appear.
- [3] Wildeman R, Dekker R, Smit ACJM. A dynamic policy for grouping maintenance activities. *European J. Operational Research* 1997; 99:530-551.
- [4] Christer AH. Developments in delay time analysis for modeling plant maintenance. *Journal of the Operational Research Society* 1999; 50:1120-1137.
- [5] Dhillon BS. A hazard rate model, *IEEE Transactions on Reliability* 1979; 28, 150-154.
- [6] Jiang R, Murthy DNP. Modeling failure-data by mixture of 2 Weibull distributions : a graphical approach. *IEEE Transactions on Reliability* 1995; 44:477-488.
- [7] Christer, AH , Wang, W. A delay-time based maintenance model for a multi-component system. *IMA Journal of Mathematics Applied in Business & Industry* 1995; 6, 205-222.
- [8] Dekker R, Van der Duyn Schouten FA, Wildeman, R. A review of multi-component maintenance models with economic dependence. *Mathematical Methods of Operations Research* 1997; 45:411-435
- [9] Murthy DNP, Nguyen DG. Study of two-component systems with failure interactions. *Naval Research Logistics Quarterly* 1985; 32:239-247
- [10] Scarf PA, Dearn M. On the development and application of maintenance policies for a two-component system with failure dependence. *IMA Journal of Mathematics Applied in Business & Industry* 1998; 9:91-107.
- [11] Berg M, Epstein B. A modified block replacement policy. *Naval Research Logistics Quarterly* 1976; 23:15-24.
- [12] Nakagawa T. *Maintenance theory of reliability*. Springer Verlag, London, 2005.
- [13] Baker RD, Scarf PA. Can models fitted to small data samples lead to maintenance policies with near-optimum cost? *IMA Journal of Mathematics Applied in Business & Industry* 1995; 6:3-12.
- [14] Scarf PA, Dwight R, Al-Musrati A. On reliability criteria and the implied cost of failure for a maintained component. *Reliability Engineering and System Safety* 2005; 89:199-207.