Top-down, bottom-up, and middle-out seasonal forecasting

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Abstract

Traditionally seasonality is estimated from an individual series. However, for a product family or a stock keeping unit (SKU) stored at different locations, estimating seasonality from the group may produce better estimates and improve forecasting accuracy. Previously, we have shown under what conditions seasonality estimated at the group level can result in better forecasting accuracy than seasonality obtained individually. This paper explores how subaggregate level forecasts can be improved by using grouping in a three-level system. Forecasts of the lowest level items can be obtained by direct forecasting from the items' history, top-down from the middle level (middle-out) and top-down from the highest level. Rules are derived in this paper to choose the best approach. Implications of the rules are discussed, particularly the role of cross correlation and how to group seasonally homogeneous series.
Keywords: seasonality, grouping, cross correlation

Introduction

Nowadays many companies are facing a demanding forecasting task for hundreds or even thousands of stock-keeping units (SKUs). These SKUs can be grouped into different product families and/or stored in different depot locations. Very often forecasts for future demands are based on short data history, because of shorter product life cycles. Demand forecasts have a significant impact on production, inventory control and service levels, and huge cost implications.

The common approach to forecast a large number of SKUs is to extrapolate each SKU's data history individually. However, Duncan *et al.* (1993) argued that "*forecasting for a particular observational unit should be more accurate if effective use is made of information, not only from a time series on that observational unit, but also from time series on similar observational units*". This paper intends to look at how forecasts can be made using data from

related items. In particular, seasonal demand forecasting will be the focus of this research. Seasonality can be estimated from the individual series itself or from a group of similar items. It is reasonable to assume that similar products from a product family or the same product kept at different locations share the same or very similar seasonal profiles. This is an area in which the use of information from relevant items may lead to better forecasts.

Some researchers have recognised the hierarchical data structure adopted by many organisations and have recommended the use of top-down or bottom-up approaches. In a top-down approach, seasonal estimates derived at a higher level of the hierarchy are employed at lower levels. In a bottom-up approach, on the other hand, seasonal estimation is conducted at the lower levels. Unfortunately, it is far from clear from the literature when to use the top-down or bottom-up forecasts.

Previous research examining the bottom-up and top-down approaches has generally been restricted to a two-level system because of the complexity of the problem. In reality, business systems with three or more levels are not uncommon. Stellwagen (2004) highlighted the need for extending current research to multiple levels. This paper derives theoretical rules to forecast seasonal demand at the SKU level by estimating seasonal indices individually, from the middle level, or from the top level in a three-level system. This theoretical perspective offers a better understanding about when to use the bottom-up, middle-out, and top-down approaches and attempts to clarify some of the confusion in the literature.

This paper is structured as follows: two group seasonal indices (GSI) methods proposed in the literature are described; rules developed for a three-level system are summarised; special cases of the rules and implications are discussed; and, finally, conclusions are drawn.

Group Seasonal Indices Methods

Two group seasonal indices (GSI) methods have been proposed in the literature: one by Dalhart (1974) and the other by Withycombe (1989).

Withycombe (1989) assumed that "*whatever causes the seasonal fluctuation in demand operates* ***the same*** *on all products within the line*" (author's own emphasis). Dalhart (1974) made the same assumption that all subaggregate series had "*a consistent underlying seasonal behaviour*". This assumption led both authors to believe that estimating seasonal indices from the group was better than from the individual series.

Dalhart (1974) proposed a group seasonal estimation method by averaging the individual seasonal indices (ISI). Let $\mathbf{S_i} = [a_{i1}, a_{i2}, \ldots a_{iq}]$, where a_{ij} is the individual seasonal index for item *i* at season j, $\mathbf{S_i}$ is the multiplicative seasonal index vector for item *i*, and *q* is the period of the seasonal cycle. Then $\mathbf{S_{DGSI}} = \frac{1}{m}\sum_{i=1}^{m}\mathbf{S_i}$, where $\mathbf{S_{DGSI}}$ is the group seasonal vector of indices estimated by Dalhart's Group Seasonal Index (DGSI) method and m is the number of series in the group. Therefore, Dalhart's method is a simple average of the individual seasonal indices.

Withycombe (1989) proposed a different method to obtain group seasonal indices (WGSI). He totalled all the series in the group and then estimated "*combined seasonal indices*" from this single time series. Therefore, Withycombe's method is a weighted average of the individual seasonal indices.

Theoretical Rules for Three Levels

<u>Models and assumptions</u>

Two models, an additive model and a mixed model, are assumed.

At the individual level:

Additive model: $Y_{ij,th} = \mu_{ij} + S_{i,h} + \varepsilon_{ij,th}$ (1)

Mixed model: $Y_{ij,th} = \mu_{ij} S_{i,h} + \varepsilon_{ij,th}$ (2)

where i = 1 to n is the suffix representing the group number at the middle level;

j = 1 to m_n is the suffix representing the item at the individual level and there are m_i items in the *i*th group;

suffix t represents the year and t = 1 to r;

suffix h represents the seasonal period and h = 1 to q;

Y represents demand;

μ_{ij} represents the underlying mean for the *j*th item in the *i*th group and is assumed to be constant over time;

$S_{i,h}$ represents a seasonal index at seasonal period h; it is unchanging from year to year and the same for all items within a group but different across groups;

$\varepsilon_{ij,th}$ is a random disturbance term for the *j*th item in the *i*th group at the *t*th year and *h*th period; it is assumed to be normally distributed with mean zero and constant variance σ_{ij}^2. There are cross correlations between any two different items both within and across groups at the same time period. Auto-correlations and cross correlations at different time periods are assumed to be zero.

Seasonality is assumed to be the same within group, but different across groups. This allows seasonal heterogeneity for different groups. We also assume stationary seasonality. (See Dekker *et al* (2004) for models and methods dealing with evolving seasonality.)

At the middle level:

Summing over m_i items in the *i*th group, the aggregate demand for the *i*th group is:

Additive model: $Y_{i,th} = \sum_{j=1}^{m_i} \mu_{ij} + m_i S_{i,h} + \sum_{j=1}^{m_i} \varepsilon_{ij,th}$ (3)

Mixed model: $Y_{i,th} = \sum_{j=1}^{m_i} \mu_{ij} S_{i,h} + \sum_{j=1}^{m_i} \varepsilon_{ij,th}$ (4)

where $Y_{i,th}$ represents the demand for the *i*th group in *t*th year and *h*th season.

At the top level:

Summing across all the groups, the demand at the top level is described by the following models:

Additive model: $Y_{th} = \sum_{i=1}^{n}\sum_{j=1}^{m_i} \mu_{ij} + \sum_{i=1}^{n} m_i S_{i,h} + \sum_{i=1}^{n}\sum_{j=1}^{m_i} \varepsilon_{ij,th}$ (5)

Mixed model: $Y_{th} = \sum_{i=1}^{n}\sum_{j=1}^{m_i} \mu_{ij} S_{i,h} + \sum_{i=1}^{n}\sum_{j=1}^{m_i} \varepsilon_{ij,th}$ (6)

where Y_{th} is the total demand in *t*th year and *h*th season.

Additive model rule

The mean squared error (MSE) for the individual seasonal index (ISI) and group seasonal index (GSI) are given by:

$$MSEISI_{ij,th} = \sigma_{ij}^2 + \frac{1}{r}\sigma_{ij}^2 \quad (7)$$

$$MSEGSI_{ij,th}^{M} = \left(1 + \frac{1}{qr}\right)\sigma_{ij}^2 + \frac{q-1}{m_i^2 qr}\sigma_i^2 \quad (8)$$

where $MSEISI_{ij,th}$ is the MSE for the *j*th item in the *i*th group in the *t*th year and *h*th season by using the ISI method, $MSEGSI_{ij,th}^{M}$ is the MSE for the *j*th item in *i*th group in the *t*th year and *h*th season by using the GSI method at the middle level, and σ_i^2 is the variance for the random terms in the *i*th group.

MSE using GSI at the top level is as follows:

$$MSEGSI_{ij,th}^{T} = \left(S_{i,h} - \frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} m_i S_{i,h} \right)^2 + \frac{1+qr}{qr}\sigma_{ij}^2 + \frac{q-1}{qr\left(\sum_{i=1}^{n} m_i\right)^2}\sigma_T^2 \qquad (9)$$

where σ_T^2 is the variance of the total demand. A detailed derivation can be found in Appendix 1.

If $S_{i,h}$ is the same as the average of the seasonal indices at the *h*th season, then $\left(S_{i,h} - \frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} m_i S_{i,h} \right)^2$ =0. The closer $S_{i,h}$ is to the average of the indices at the same period, the smaller the $\left(S_{i,h} - \frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} m_i S_{i,h} \right)^2$ term is.

It follows that

$MSEGSI_{ij,th}^{T} < MSEISI_{ij,th}$ if and only if:

$$\left(S_{i,h} - \frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} m_i S_{i,h} \right)^2 < \frac{q-1}{qr}\left[\sigma_{ij}^2 - \frac{\sigma_T^2}{\left(\sum_{i=1}^{n} m_i\right)^2} \right] \qquad (10)$$

Inequality (10) shows that the choice between ISI and GSI at the top level depends, in general, on two factors: seasonal homogeneity to the group and noisiness of the series. The term $\sigma_{ij}^2 - \frac{\sigma_T^2}{\left(\sum_{i=1}^{n} m_i\right)^2}$ on the right-side of the inequality (10) reveals that if the noisiness of the individual item σ_{ij}^2 is less

than the "average" of the total $\frac{\sigma_T^2}{\left(\sum_{i=1}^{n} m_i\right)^2}$, the right-side of the inequality would be negative. However, the left-side of the inequality is either zero (seasonal homogeneity) or positive (seasonal heterogeneity). This means if the individual noisiness is less than the "average" of the total, ISI is always better than GSI despite seasonal patterns. Seasonal homogeneity becomes relevant only when the individual noisiness is greater than the "average" of the total.

Strictly speaking, inequality (10) applies for a single season *h*. Theoretically justified, it is operationally cumbersome to compare ISI and GSI for each season. An alternative is to compare the two methods summed over a year:

$$\frac{1}{q}\sum_{h=1}^{q}\left(S_{i,h} - \frac{1}{\sum_{i=1}^{n} m_i}\sum_{i=1}^{n} m_i S_{i,h}\right)^2 < \frac{q-1}{qr}\left[\sigma_{ij}^2 - \frac{\sigma_T^2}{\left(\sum_{i=1}^{n} m_i\right)^2}\right] \qquad (11)$$

Inequality (11) suggests that if individual noisiness is less than the "average" of the total, ISI is better than GSI without the need to consider seasonal profiles. However, if this is not the case, seasonal homogeneity should be considered. If seasonality of the target series is very different from the rest of the series, it is better to use ISI as $\frac{1}{q}\sum_{h=1}^{q}\left(S_{i,h} - \frac{1}{\sum_{i=1}^{n} m_i}\sum_{i=1}^{n} m_i S_{i,h}\right)^2$ would be large. It makes sense intuitively that GSI would be useful if the individual item's seasonal pattern is homogeneous to the group so that it can "borrow strength" from the group.

Similarly, the following inequality compares GSI from the top level with GSI from the middle level:

$MSEGSI_{ij,th}^{T} < MSEGSI_{ij,th}^{M}$ if and only if:

$$\left(S_{i,h} - \frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} m_i S_{i,h} \right)^2 < \frac{q-1}{qr} \left[\frac{\sigma_i^2}{m_i^2} - \frac{\sigma_T^2}{\left(\sum_{i=1}^{n} m_i \right)^2} \right] \qquad (12)$$

To choose at which level GSI should be applied depends on the factors of seasonal homogeneity to the group and noisiness. Since seasonality is assumed to be the same within group at the middle level, the cross-group seasonal homogeneity is very important to decide whether GSI from the top level is better than GSI from the middle level. Another factor is the "average" noisiness of the groups (either at middle level or at top level). If $\frac{\sigma_i^2}{m_i^2} < \frac{\sigma_T^2}{\left(\sum_{i=1}^{n} m_i \right)^2}$, then GSI from the middle level is better even when seasonal homogeneity across groups is satisfied. If $\frac{\sigma_i^2}{m_i^2} > \frac{\sigma_T^2}{\left(\sum_{i=1}^{n} m_i \right)^2}$, then GSI from the top level is better.

Over a whole year, inequality (12) becomes:

$$\frac{1}{q} \sum_{h=1}^{q} \left(S_{i,h} - \frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} m_i S_{i,h} \right)^2 < \frac{q-1}{qr} \left[\frac{\sigma_i^2}{m_i^2} - \frac{\sigma_T^2}{\left(\sum_{i=1}^{n} m_i \right)^2} \right] \qquad (13)$$

<u>Mixed model rule</u>

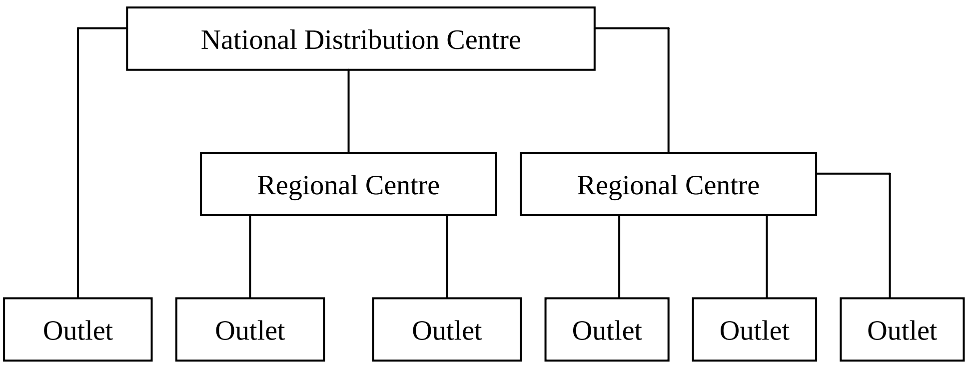
When the mixed model is assumed, the comparison is between ISI, DGSI and WGSI.

$$MSEISI_{ij,th} = \left(1 + \frac{1}{r}\right)\sigma_{ij}^2 \tag{14}$$

$$MSEDGSI_{ij,th}^{M} = \sigma_{ij}^2 + \frac{\mu_{ij}^2 \sigma_{i1}^2}{m_i^2 r \mu_{11}^2} + \ldots + \frac{\mu_{ij}^2 \sigma_{im_i}^2}{m_i^2 r \mu_{im_i}^2} + \frac{2\mu_{ij}^2}{m_i^2 r} \sum_{k=1}^{m_i - 1} \sum_{l=k+1}^{m_i} \frac{\rho_{ik,il} \sigma_{ik} \sigma_{il}}{\mu_{ik} \mu_{il}} \tag{15}$$

$$MSEWGSI_{ij,th}^{M} = \sigma_{ij}^2 + \frac{\mu_{ij}^2}{r \mu_A^2} \left(\sigma_{i1}^2 + \sigma_{i2}^2 + \ldots + \sigma_{im_i}^2 + 2 \sum_{k=1}^{m_i - 1} \sum_{l=j+1}^{m_i} \rho_{ik,il} \sigma_{ik} \sigma_{il} \right) \tag{16}$$

DGSI at the top level can be calculated in two ways: from the individual level, averaging all the ISIs from each individual series; or from the middle level, averaging the DGSIs from the groups. We decided to calculate DGSI at the top level from the individual level. The reason is that the middle level may not always exist for all items. The following diagram illustrates the point:



The above example of geographical grouping shows that in the case of an end-customer in the supply chain supplied directly by the national distribution centre, there is not a middle level, i.e. the regional centre for that particular outlet. Therefore, it is not always possible to calculate DGSI for the top level from the middle level; but it can always be done from the individual level.

$$MSEDGSI_{ij,th}^{T}=\mu_{ij}^{2}\left(S_{i,h}-\frac{\sum_{i=1}^{n}m_{i}S_{i,h}}{\sum_{i=1}^{n}m_{i}}\right)^{2}+\sigma_{ij}^{2}+\frac{\mu_{ij}^{2}}{r\left(\sum_{i=1}^{n}m_{i}\right)^{2}}\left(\frac{\sigma_{11}^{2}}{\mu_{11}^{2}}+\ldots+\frac{\sigma_{nm_{n}}^{2}}{\mu_{nm_{n}}^{2}}\right)$$
$$+\frac{2}{r\left(\sum_{i=1}^{n}m_{i}\right)^{2}}\sum_{i=1}^{n}\sum_{k=1}^{m_{i}-1}\sum_{l=k+1}^{m_{i}}\frac{\mu_{ik}^{2}}{\mu_{ik}\mu_{il}}\rho_{ik,il}\sigma_{ik}\sigma_{il}+\frac{2}{r\left(\sum_{i=1}^{n}m_{i}\right)^{2}}\sum_{i=1}^{n-1}\sum_{u=i+1}^{n}\sum_{k=1}^{m_{i}}\sum_{v=1}^{m_{u}}\frac{\mu_{ik}^{2}}{\mu_{ik}\mu_{il}}\rho_{ik,uv}\sigma_{ik}\sigma_{uv} \tag{17}$$

$MSEDGSI_{ij,(r+1)h}^{T}<MSEISI_{ij,(r+1)h}$ if and only if:

$$\left(S_{i,h}-\frac{\sum_{i=1}^{n}m_{i}S_{i,h}}{\sum_{i=1}^{n}m_{i}}\right)^{2}<\frac{1}{r}\left[\frac{\sigma_{ij}^{2}}{\mu_{ij}^{2}}-\left(\begin{array}{l}\frac{1}{\left(\sum_{i=1}^{n}m_{i}\right)^{2}}\left(\frac{\sigma_{11}^{2}}{\mu_{11}^{2}}+\ldots+\frac{\sigma_{nm_{n}}^{2}}{\mu_{nm_{n}}^{2}}\right)+\frac{2}{\left(\sum_{i=1}^{n}m_{i}\right)^{2}}\sum_{i=1}^{n}\sum_{k=1}^{m_{i}-1}\sum_{l=k+1}^{m_{i}}\frac{\rho_{ik,il}\sigma_{ik}\sigma_{il}}{\mu_{ik}\mu_{il}}\\+\frac{2}{r\left(\sum_{i=1}^{n}m_{i}\right)^{2}}\sum_{i=1}^{n-1}\sum_{u=i+1}^{n}\sum_{k=1}^{m_{i}}\sum_{v=1}^{m_{u}}\frac{\rho_{ik,uv}\sigma_{ik}\sigma_{uv}}{\mu_{ik}\mu_{uv}}<\left(1+\frac{1}{r}\right)\sigma_{ij}^{2}\end{array}\right)\right] \tag{18}$$

The comparison between DGSI from the top level and ISI is similar to the comparison in the additive model. The dominating factor is the noisiness of the individual series and the "average" noisiness of the total. In the mixed model, noisiness is measured by coefficient of variation rather than variance as in the additive model. If the individual noisiness is less than the group average, then ISI is better than DGSI from the top level without considering seasonal homogeneity. However, if individual noisiness is greater, then seasonal homogeneity should be examined. The measure of seasonal homogeneity is the same expression as in the additive model.

Inequality (18) can be summed over a year for operational purposes:

$$\frac{1}{q}\sum_{h=1}^{q}\left(S_{i,h}-\frac{1}{\sum_{i=1}^{n}m_i}\sum_{i=1}^{n}m_iS_{i,h}\right)^2<\frac{1}{r}\left[\frac{\sigma_{ij}^2}{\mu_{ij}^2}-\left(\begin{array}{l}\frac{1}{\left(\sum_{i=1}^{n}m_i\right)^2}\left(\frac{\sigma_{11}^2}{\mu_{11}^2}+\ldots+\frac{\sigma_{nm_n}^2}{\mu_{nm_n}^2}\right)+\frac{2}{\left(\sum_{i=1}^{n}m_i\right)^2}\sum_{i=1}^{n}\sum_{k=1}^{m_i-1}\sum_{l=k+1}^{m_i}\frac{\rho_{ik,il}\sigma_{ik}\sigma_{il}}{\mu_{ik}\mu_{il}}\\+\frac{2}{r\left(\sum_{i=1}^{n}m_i\right)^2}\sum_{i=1}^{n-1}\sum_{u=i+1}^{n}\sum_{k=1}^{m_i}\sum_{v=1}^{m_u}\frac{\rho_{ik,uv}\sigma_{ik}\sigma_{uv}}{\mu_{ik}\mu_{uv}}\end{array}\right)\right]$$

(19)

The rule to compare DGSI from top level and middle level is as follows:

$MSEDGSI_{ij,(r+1)h}^{T}<MSEDGSI_{ij,(r+1)h}^{M}$ if and only if:

$$\left(S_{i,h}-\frac{\sum_{i=1}^{n}m_iS_{i,h}}{\sum_{i=1}^{n}m_i}\right)^2<\left[\frac{1}{rm_i^2}\left(\frac{\sigma_{i1}^2}{\mu_{11}^2}+\ldots\frac{\sigma_{im_i}^2}{\mu_{im_i}^2}+2\sum_{k=1}^{m_i-1}\sum_{l=k+1}^{m_i}\frac{\rho_{ik,il}\sigma_{ik}\sigma_{il}}{\mu_{ik}\mu_{il}}\right)-\frac{1}{r\left(\sum_{i=1}^{n}m_i\right)^2}\left(\begin{array}{l}\frac{\sigma_{11}^2}{\mu_{11}^2}+\ldots+\frac{\sigma_{nm_n}^2}{\mu_{nm_n}^2}\\+2\sum_{i=1}^{n}\sum_{k=1}^{m_i-1}\sum_{l=k+1}^{m_i}\frac{\rho_{ik,il}\sigma_{ik}\sigma_{il}}{\mu_{ik}\mu_{il}}\\+2\sum_{i=1}^{n-1}\sum_{u=i+1}^{n}\sum_{k=1}^{m_i}\sum_{v=1}^{m_u}\frac{\rho_{ik,uv}\sigma_{ik}\sigma_{uv}}{\mu_{ik}\mu_{uv}}\end{array}\right)\right]$$

(20)

The comparison mainly depends on the “average” noisiness from the middle level and from the top level. For DGSI from the top level to be better, the “average” noisiness from the top level has to be less than that from the middle level and $\left(S_{i,h}-\frac{\sum_{i=1}^{n}m_iS_{i,h}}{\sum_{i=1}^{n}m_i}\right)^2$ less than the difference of the noisiness from the two levels.

Over a whole year:

$$\frac{1}{q}\sum_{h=1}^{q}\left(S_{i,h}-\frac{1}{\sum_{i=1}^{n}m_i}\sum_{i=1}^{n}m_iS_{i,h}\right)^2<\left[\frac{1}{rm_i^2}\left(\frac{\sigma_{i1}^2}{\mu_{11}^2}+\ldots\frac{\sigma_{im_i}^2}{\mu_{im_i}^2}+2\sum_{k=1}^{m_i-1}\sum_{l=k+1}^{m_i}\frac{\rho_{ik,il}\sigma_{ik}\sigma_{il}}{\mu_{ik}\mu_{il}}\right)-\frac{1}{r\left(\sum_{i=1}^{n}m_i\right)^2}\left(\begin{array}{l}\frac{\sigma_{11}^2}{\mu_{11}^2}+\ldots+\frac{\sigma_{nm_n}^2}{\mu_{nm_n}^2}\\+2\sum_{i=1}^{n}\sum_{k=1}^{m_i-1}\sum_{l=k+1}^{m_i}\frac{\rho_{ik,il}\sigma_{ik}\sigma_{il}}{\mu_{ik}\mu_{il}}\\+2\sum_{i=1}^{n-1}\sum_{u=i+1}^{n}\sum_{k=1}^{m_i}\sum_{v=1}^{m_u}\frac{\rho_{ik,uv}\sigma_{ik}\sigma_{uv}}{\mu_{ik}\mu_{uv}}\end{array}\right)\right]$$

(21)

The rules comparing WGSI at different levels are presented as follows:

$MSEWGSI^{T}_{ij,(r+1)h} < MSEISI_{ij,(r+1)h}$ if and only if:

$$\left(S_{i,h} - \frac{\sum_{i=1}^{n}\sum_{j=1}^{m_i} \mu_{ij} S_{i,h}}{\mu_A} \right)^2 < \frac{1}{r}\left(\frac{\sigma_{ij}^2}{\mu_{ij}^2} - \frac{\sigma_A^2}{\mu_A^2} \right) \tag{22}$$

$MSEWGSI^{T}_{ij,(r+1)h} < MSEWGSI^{M}_{ij,(r+1)}$ if and only if:

$$\left(S_{i,h} - \frac{\sum_{i=1}^{n}\sum_{j=1}^{m_i} \mu_{ij} S_{i,h}}{\mu_A} \right)^2 < \frac{1}{r}\left(\frac{\sigma_{i}^2}{\mu_{i}^2} - \frac{\sigma_A^2}{\mu_A^2} \right) \tag{23}$$

Inequalities (22) and (23) can be summarised over a whole year as well to cover the complete seasonal cycle.

The term on the left-hand side of (22) and (23) is different from the term developed for the additive model and for DGSI. This is because WGSI is calculated from the aggregate series.

A full derivation of the rules can be found in Appendix 2. The rules are developed for a three-level system; however, they can be extended to any number of levels.

Special Cases of the Rules

A special case of the general rules developed for a three-level system can be applied for two levels. The special case still allows seasonal heterogeneity among series and has not been examined previously.

Additive model:

The following inequality (24) can determine when to use GSI:

$$\frac{1}{q}\sum_{h=1}^{q}\left(S_{i,h}-\frac{1}{m}\sum_{i=1}^{m}S_{i,h}\right)^2<\frac{q-1}{qr}\left[\sigma_i^2-\frac{\sigma_T^2}{m^2}\right] \quad (24)$$

where m is the number of items in a group.

Mixed model:

Use DGSI instead of ISI if and only if:

$$\frac{1}{q}\sum_{h=1}^{q}\left(S_{i,h}-\frac{1}{m}\sum_{i=1}^{m}S_{i,h}\right)^2<\frac{1}{r}\left[\frac{\sigma_i^2}{\mu_i^2}-\frac{1}{m^2}\left(\sum_{i=1}^{m}\frac{\sigma_i^2}{\mu_i^2}+2\sum_{j=1}^{m-1}\sum_{l=j+1}^{m}\frac{1}{\mu_j}\frac{1}{\mu_l}\rho_{jl}\sigma_j\sigma_l\right)\right] \quad (25)$$

Use WGSI instead of ISI if and only if:

$$\frac{1}{q}\sum_{h=1}^{q}\left(S_{i,h}-\frac{\sum_{i=1}^{m}\mu_i S_{i,h}}{\mu_A}\right)^2<\frac{1}{r}\left(\frac{\sigma_i^2}{\mu_i^2}-\frac{\sigma_1^2+\ldots+\sigma_m^2+2\sum_{j=1}^{m-1}\sum_{l=j+1}^{m}\rho_{jl}\sigma_j\sigma_l}{\mu_A^2}\right) \quad (26)$$

Full derivations of the rules can be found in Appendix 3. The theoretical rules for two levels developed previously depended only on the noisiness of the individual series and the "average" noisiness of the group. Seasonality did not play a role in the rules because it was assumed that there was common seasonality within the group.

Inequalities (24) to (26) show that noisiness is still the dominating factor, and if the noisiness of the individual series is less than the "average" of the group, ISI should be used regardless of the seasonal factors. If individual series is noisier than the "average" of the group, then it might borrow strength from the group and seasonal homogeneity is important. These rules clearly show that the grouping approach would still be beneficial even a certain degree of seasonal heterogeneity is allowed. But more importantly, the rules show in what form seasonal homogeneity affects the choice between ISI and GSI.

Inequality (24) can be re-arranged as:

$$r < \frac{(q-1)\left(\sigma_i^2 - \frac{\sigma_T^2}{m^2}\right)}{\sum_{h=1}^{q}\left(S_{i,h} - \frac{1}{m}\sum_{i=1}^{m} S_{i,h}\right)^2} \tag{27}$$

Inequality (27) shows that the number of years of data history (r) affects the choice between ISI and GSI. When there is seasonal heterogeneity but GSI is still preferred, the number of years of data has to be less than the right-hand side of inequality (27). It makes sense that, when data history is short, ISI is unlikely to be accurate. Therefore, GSI is better even allowing some degree of heterogeneity. When data history becomes longer, the estimation of ISI will be more accurate. This is also true for the mixed model. If seasonality is homogeneous, then r is irrelevant as the choice only depends on the noisiness of the individual series and the "average" noisiness from the group.

Implications of the Rules

Seasonal Grouping Mechanism

An important implication of the rules developed in this paper is that $\frac{1}{q}\sum_{h=1}^{q}\left(S_{i,h} - \frac{1}{\sum_{i=1}^{n} m_i}\sum_{i=1}^{n} m_i S_{i,h}\right)^2$ and $\frac{1}{q}\sum_{h=1}^{q}\left(S_{i,h} - \frac{\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mu_{ij} S_{i,h}}{\mu_A}\right)^2$ are theoretically informed expressions that can be used as distance measures to define seasonal groups. How to group seasonally homogeneous series is a very important research question in this area, recognised by both Dalhart (1974) and Withycombe (1989). Company definition may not always result in seasonal homogeneity within groups. Although there have been some early papers in the literature looking at grouping/classification methods for inventory purposes (Flores and Whybark, 1986, 1987; Maier and Shimkin, 1988; Cohen and Ernst, 1988), there was no theoretical work directly addressing the problem of forming seasonally homogeneous groups. Some researchers

proposed to use cluster analysis. Bunn and Vassilopoulos (1993) used cluster analysis with Euclidean distances to define seasonal groups. It was documented in that paper and Vassilopoulos (1994) that cluster analysis was applied to define seasonally homogeneous groups. The software package SPSS/PC+ was used and "*the average linkage between-groups method was implemented to join clusters and the Euclidean distance to measure nearness*" (Vassilopoulos, 1994). Assume there are *m* series in group *i* and *n* series in group *j*, the grouping mechanism using the average linkage between groups joins groups which minimises $\frac{1}{mnq}\sum_{h=1}^{q}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(S_{i,h}-S_{j,h}\right)^2$. There are various distance measures for clustering in SPSS. This measure was chosen arbitrarily and the authors commented that it resulted in seasonally homogeneous and distinct groups.

The expressions derived in this paper are theory informed. They are equivalent to the k-means clustering (available in SPSS), which assigns each point to the cluster whose centre (also called centroid) is the nearest. Both expressions are in a Euclidean distance form. The key difference is that the average linkage between-groups method used by Bunn and Vassilopoulos (1993) and Vassilopoulos (1994) is a pair-wise comparison for all seasonal indices over a seasonal cycle; while the measure derived in this paper compares a seasonal index to the centre of gravity.

Role of Cross Correlation

Another important implication of the rules is the clarification of the role of correlation. Previous research on the issue of grouping has consistently suggested cross correlation as the most important factor to decide whether a direct forecast or a derived (top-down or bottom-up) forecast should be used. However, there have been arguments on whether series with positive or negative correlation should be grouped.

Duncan *et al.* (1998) claimed that analogous series should correlate positively (co-vary) over time and argued that the co-variation will be able to "*add precision to model estimates and to adapt quickly to time-series pattern changes*".

Schwarzkopf *et al.* (1988) maintained that the most significant claim for top-down analysis is that it is inherently more accurate than bottom-up. This is supported by the statistical principle that the average of a number of items is less variable than the individual items. They also acknowledged the importance of cross correlation in the comparison between the direct and the top-down forecasts. "*If there is a strong positive correlation in demand for items in a group, the variance for the family is increased by the amount of the covariance term*". Then the aggregate is no longer always better than the bottom-up.

Previous research in the literature did not distinguish between a common model and varied models, thus causing much confusion. Chen and Boylan (2007) established formally that, given the same model and seasonal homogeneity within groups, it was negative correlation between series that favoured the top-down approach.

Inequalities (24) to (26) offer further insights on this issue when series follow different models because of seasonal heterogeneity.

Let us suppose $Y_{i,th} = \mu_i + S_{i,h} + \varepsilon_{i,th}$ and $Y_{j,th} = \mu_j + S_{j,h} + \varepsilon_{j,th}$.

Then

$$\operatorname{cov}\left(Y_{i,th}Y_{j,th}\right) = E\frac{1}{q}\sum_{h=1}^{q}\left(\mu_i + S_{i,h} + \varepsilon_{i,th}\right)\left(\mu_j + S_{j,h} + \varepsilon_{j,th}\right)$$

$$-E\frac{1}{q}\sum_{h=1}^{q}\left(\mu_i + S_{i,h} + \varepsilon_{i,th}\right)E\frac{1}{q}\sum_{h=1}^{q}\left(\mu_j + S_{j,h} + \varepsilon_{j,th}\right)$$

$$= E\frac{1}{q}\sum_{h=1}^{q}\left(\mu_i\mu_j + S_{i,h}\mu_j + \varepsilon_{i,th}\mu_j + \mu_i S_{j,h} + S_{i,h}S_{j,h} + \varepsilon_{i,th}S_{j,h} + \mu_i\varepsilon_{j,th} + S_{i,h}\varepsilon_{j,th} + \varepsilon_{i,th}\varepsilon_{j,th}\right) - \mu_i\mu_j$$

$$= \frac{1}{q}\sum_{h=1}^{q} S_{i,h}S_{j,h} + E\left(\varepsilon_{i,th}\varepsilon_{j,th}\right) \qquad (28)$$

Since seasonal heterogeneity is allowed here, the two series i and j follow different models. It is obvious from equation (28) that two components contribute towards the overall covariance between the two series: covariance between the random errors and the "co-movement" of the seasonals. If one series has a positive seasonality and the other has a negative one, then this would affect the covariance (and correlation) between the two series. Previous research did not identify how seasonal heterogeneity and correlation between random errors affect the overall correlation between two series. The debate about grouping series with positive or negative correlation was looking at the wrong question. On the other hand, our rules, both for the general rules and special cases, clearly demonstrate how seasonal heterogeneity and noisiness of series, which include the term of cross correlation between random errors, affect the choice between ISI and GSI.

Conclusions and Further Research

This paper has established theoretical rules to choose the best approach for subaggregate seasonal demand forecasting in a three-level system. Seasonality can be estimated from the series itself (bottom-up), from a group of similar items (middle-out), or from all items with or without similar seasonality (top-down). The rules can be easily extended to any level following the same argument.

A special case of the rules is applied on two levels. This also allows a certain degree of seasonal heterogeneity among series. The results are intuitive.

The expressions $\frac{1}{q}\sum_{h=1}^{q}\left(S_{i,h}-\frac{1}{\sum_{i=1}^{n}m_i}\sum_{i=1}^{n}m_i S_{i,h}\right)^2$ and $\frac{1}{q}\sum_{h=1}^{q}\left(S_{i,h}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mu_{ij}S_{i,h}}{\mu_A}\right)^2$ derived theoretically can be used as distance measures to define seasonal grouping. Previous researchers have recognised grouping mechanism as a very important research question but no one attempted to solve this. The clustering method used by Bunn and Vassilopoulos (1993) lacked any theoretical justification, although providing satisfactory results in their empirical experiment. The expressions we developed are theory informed and are of a Euclidian form.

The rules are important from a theoretical perspective, and they can be easily operationalised. Equally important are the insights gained from the rules. They provide a better understanding about how seasonal heterogeneity, noisiness of series, and cross correlation affect the choice of different approaches. The findings clarify some of the confusion about the role of cross correlation.

For further research simulations will be conducted to compare the theory informed grouping mechanisms to other methods, e.g. the clustering method used by Bunn and Vassilopoulos (1993). Various algorithms, e.g. joining single items into groups or starting as one big cluster and then dividing items into smaller groups, will be used with the distances measures on simulated data. Results will be tested on real data to examine the robustness of the rules.

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Appendix 1

The Additive Model

Individual level:

$$Y_{ij,th} = \mu_{ij} + S_{i,h} + \varepsilon_{ij,th} \tag{1}$$

Middle level:

Summing over m_i items in the *i*th group, the aggregate demand for the *i*th group is:

$$Y_{i,th} = \sum_{j=1}^{m_i} \mu_{ij} + m_i S_{i,h} + \sum_{j=1}^{m_i} \varepsilon_{ij,th} \tag{2}$$

where $Y_{i,th}$ represents the demand for the *i*th group in *t*th year and *h*th season.

Top level:

Summing across all the groups, the demand at the top level is described by the following model:

$$Y_{th} = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \mu_{ij} + \sum_{i=1}^{n} m_i S_{i,h} + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \varepsilon_{ij,th} \tag{3}$$

where Y_{th} is the total demand in *t*th year and *h*th season.

Following the same argument as in Chen and Boylan (2007):

$$MSEISI_{ij,(r+1)h} = \sigma_{ij}^2 + \frac{1}{r}\sigma_{ij}^2 \tag{4}$$

$$MSEGSI_{ij,(r+1)h}^{M} = \left(1 + \frac{1}{qr}\right)\sigma_{ij}^2 + \frac{q-1}{m_i^2 qr}\sigma_i^2 \tag{5}$$

where $MSEISI_{ij,th}$ is the mean squared error (MSE) for the *j*th item in the *i*th group in the *(r+1)*th year and *h*th season by using the individual seasonal indices (ISI) method, $MSEGSI_{ij,th}^{M}$ is the MSE for the *j*th item in *i*th group in the *(r+1)*th year and *h*th season by using the group seasonal indices (GSI) method at the middle level, and σ_i^2 is the variance for the random terms in the *i*th group.

The MSE expression by using the GSI method at the top level is derived as follows:

$$Y_{th} = \sum_{i=1}^{n}\sum_{j=1}^{m_i}\mu_{ij} + \sum_{i=1}^{n} m_i S_{i,h} + \sum_{i=1}^{n}\sum_{j=1}^{m_i}\varepsilon_{ij,th} = \mu + S_h + \varepsilon_{th} \qquad (6)$$

$$Y_{th} = \mu + S_h + \varepsilon_{th} \sim N\left(\mu + S_h, \sigma^2\right) \text{for a given h}$$
$$f\left(Y_{th}\right) = f\left(\mu + S_h + \varepsilon_{th}\right)$$
$$L\left(\mu, S_h, \sigma^2 \mid Y_{1h}, Y_{2h}, \ldots Y_{rh}\right)$$
$$= \prod_{t=1}^{r}\left\{\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{Y_{th} - \mu - S_h}{\sigma}\right)^2\right]\right\}$$

The log - likelihood function is given by :

$$\log L = \sum_{t=1}^{r}\left\{\log\left(\frac{1}{\sqrt{2\pi}\sigma}\right) - \frac{1}{2}\left(\frac{Y_{th} - \mu - S_h}{\sigma}\right)^2\right\}$$
$$= -\frac{1}{2} r\log(2\pi) - \frac{1}{2} r\log\left(\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{t=1}^{r}\left(Y_{th} - \mu - S_h\right)^2$$

Hence, differentiating with respect to $\left(\mu + S_h\right)$,

$$\frac{\partial \log L}{\partial\left(\mu + S_h\right)} = \frac{1}{\sigma^2}\left(\sum_{t=1}^{r} Y_{th} - r\mu - rS_h\right) = 0 \qquad (7)$$

Re-arranging (7),

$$\hat{\mu} + \hat{S}_h = \frac{1}{r}\sum_{t=1}^{r} Y_{th} \qquad (8)$$

Summing over all the seasons in the year and requiring that the seasonal indices sum to zero,

$$\hat{\mu} = \frac{1}{qr}\sum_{t=1}^{r}\sum_{h=1}^{q} Y_{th} \qquad (9)$$

Hence, by substitution:

$$\hat{S}_h = \frac{1}{r}\sum_{t=1}^{r} Y_{th} - \frac{1}{qr}\sum_{t=1}^{r}\sum_{h=1}^{q} Y_{th} \qquad (10)$$

Disaggregate (10) and $GSI_h = \frac{1}{r\sum_{i=1}^{m} m_i}\sum_{t=1}^{r} Y_{th} - \frac{1}{qr\sum_{i=1}^{m} m_i}\sum_{t=1}^{r}\sum_{h=1}^{q} Y_{th}$ (11)

where $\sum_{i=1}^{m} m_i$ is the total number of items. (11) is the GSI method obtained from the top level.

$$MSEGSI_{ij,(r+1)h}^{T} = E\left(Y_{ij,(r+1)h} - \frac{1}{qr}\sum_{t=1}^{r}\sum_{h=1}^{q} Y_{ij,th} - \frac{1}{r\sum_{i=1}^{m} m_i}\sum_{t=1}^{r} Y_{th} + \frac{1}{qr\sum_{i=1}^{m} m_i}\sum_{t=1}^{r}\sum_{h=1}^{q} Y_{th} \right)^2$$

=

$$E\begin{bmatrix} \mu_{ij} + S_{i,h} + \varepsilon_{ij,(r+1)h} - \mu_{ij} - \frac{1}{qr}\sum_{t=1}^{r}\sum_{h=1}^{q}\varepsilon_{ij,th} - \frac{1}{r\sum_{i=1}^{n} m_i}\sum_{t=1}^{r}\left(\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mu_{ij} + \sum_{i=1}^{n} m_i S_{i,h} + \sum_{i=1}^{n}\sum_{j=1}^{m_i}\varepsilon_{ij,th} \right) \\ + \frac{1}{qr\sum_{i=1}^{n} m_i}\sum_{t=1}^{r}\sum_{h=1}^{q}\left(\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mu_{ij} + \sum_{i=1}^{n} m_i S_{i,h} + \sum_{i=1}^{n}\sum_{j=1}^{m_i}\varepsilon_{ij,th} \right) \end{bmatrix}^2$$

$$= E\begin{pmatrix} S_{i,h} + \varepsilon_{ij,(r+1)h} - \frac{1}{qr}\sum_{t=1}^{r}\sum_{h=1}^{q}\varepsilon_{ij,th} - \frac{1}{\sum_{i=1}^{n} m_i}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mu_{ij} - \frac{1}{\sum_{i=1}^{n} m_i}\sum_{i=1}^{n} m_i S_{i,h} \\ - \frac{1}{r\sum_{i=1}^{n} m_i}\sum_{t=1}^{r}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\varepsilon_{ij,th} + \frac{1}{\sum_{i=1}^{n} m_i}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mu_{ij} + \frac{1}{qr\sum_{i=1}^{n} m_i}\sum_{t=1}^{r}\sum_{h=1}^{q}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\varepsilon_{ij,th} \end{pmatrix}^2$$

$$= S_{i,h}^2 + \sigma_{ij}^2 + \frac{1}{qr}\sigma_{ij}^2 + \left(\frac{1}{\sum_{i=1}^{n} m_i}\sum_{i=1}^{n} m_i S_{i,h} \right)^2 + \frac{1}{r\left(\sum_{i=1}^{n} m_i\right)^2}\sigma_T^2 + \frac{1}{qr\left(\sum_{i=1}^{n} m_i\right)^2}\sigma_T^2$$

$$- \frac{2S_{i,h}}{\sum_{i=1}^{n} m_i}\sum_{i=1}^{n} m_i S_{i,h} + \frac{2}{qr^2\sum_{i=1}^{n} m_i}\left[r\sigma_{ij}^2 + rE\left(\varepsilon_{ij,th}\sum_{k\neq i}\sum_{l\neq j}\varepsilon_{kl,th} \right) \right]$$

$$- \frac{2}{q^2 r^2\sum_{i=1}^{n} m_i}\left[qr\sigma_{ij}^2 + qrE\left(\varepsilon_{ij,th}\sum_{k\neq i}\sum_{l\neq j}\varepsilon_{kl,th} \right) \right] - \frac{2}{qr\left(\sum_{i=1}^{n} m_i\right)^2}\sigma_T^2$$

$$= S_{i,h}^{2} - \frac{2S_{i,h}}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} m_i S_{i,h} + \left(\frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} m_i S_{i,h} \right)^2 + \frac{1+qr}{qr} \sigma_{ij}^2 + \frac{q-1}{qr\left(\sum_{i=1}^{n} m_i\right)^2} \sigma_T^2$$

Therefore, $$MSEGSI_{ij,(r+1)h}^{T} = \left(S_{i,h} - \frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} m_i S_{i,h} \right)^2 + \frac{1+qr}{qr} \sigma_{ij}^2 + \frac{q-1}{qr\left(\sum_{i=1}^{n} m_i\right)^2} \sigma_T^2 \quad (12)$$

$MSEGSI_{ij,(r+1)h}^{T} < MSEISI_{ij,(r+1)h}$ if and only if:

$$\left(S_{i,h} - \frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} m_i S_{i,h} \right)^2 + \frac{1+qr}{qr} \sigma_{ij}^2 + \frac{q-1}{qr\left(\sum_{i=1}^{n} m_i\right)^2} \sigma_T^2 < \sigma_{ij}^2 + \frac{1}{r} \sigma_{ij}^2$$

$$\left(S_{i,h} - \frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} m_i S_{i,h} \right)^2 < \frac{q-1}{qr} \left[\sigma_{ij}^2 - \frac{\sigma_T^2}{\left(\sum_{i=1}^{n} m_i\right)^2} \right] \quad (13)$$

$MSEGSI_{ij,(r+1)h}^{T} < MSEISI_{ij,(r+1)h}^{M}$ if and only if:

$$\left(S_{i,h} - \frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} m_i S_{i,h} \right)^2 + \frac{1+qr}{qr} \sigma_{ij}^2 + \frac{q-1}{qr\left(\sum_{i=1}^{n} m_i\right)^2} \sigma_T^2 < \frac{1+qr}{qr} \sigma_{ij}^2 + \frac{q-1}{m_i^2 qr} \sigma_i^2$$

$$\left(S_{i,h} - \frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} m_i S_{i,h} \right)^2 < \frac{q-1}{qr} \left[\frac{\sigma_i^2}{m_i^2} - \frac{\sigma_T^2}{\left(\sum_{i=1}^{n} m_i\right)^2} \right] \quad (14)$$

Appendix 2

The Mixed Model

Individual level: $Y_{ij,th} = \mu_{ij} S_{i,h} + \varepsilon_{ij,th}$ (15)

Middle level: $Y_{i,th} = \sum_{j=1}^{m_i} \mu_{ij} S_{i,h} + \sum_{j=1}^{m_i} \varepsilon_{ij,th}$ (16)

Top level: $Y_{th} = \sum_{i=1}^{n}\sum_{j=1}^{m_i} \mu_{ij} S_{i,h} + \sum_{i=1}^{n}\sum_{j=1}^{m_i} \varepsilon_{ij,th}$ (17)

Follow the same argument as in Chen and Boylan (2007):

$$MSEISI_{ij,(r+1)h} = \left(1+\frac{1}{r}\right)\sigma_{ij}^2 \tag{18}$$

$$MSEDGSI^{M}_{ij,(r+1)h} = \sigma_{ij}^2 + \frac{\mu_{ij}^2\sigma_{i1}^2}{\mu_{11}^2 m_i^2 r} + \ldots + \frac{\mu_{ij}^2\sigma_{im_i}^2}{\mu_{im_i}^2 m_i^2 r} + \frac{2\mu_{ij}^2}{m_i^2 r}\sum_{k=1}^{m_i-1}\sum_{l=k+1}^{m_i}\frac{\rho_{ik,il}\sigma_{ik}\sigma_{il}}{\mu_{ik}\mu_{il}} \tag{19}$$

$$DGSI_h = \frac{ISI_{11} + \ldots + ISI_{1m_1} + \ldots + ISI_{nm_n}}{\sum_{i=1}^{n} m_i}$$

$$= \frac{\dfrac{Y_{11,1h} + Y_{11,2h} + \ldots + Y_{11,rh}}{r\hat{\mu}_{11}} + \ldots + \dfrac{Y_{nm_n,1h} + Y_{nm_n,2h} + \ldots + Y_{nm_n,rh}}{r\hat{\mu}_{nm_n}}}{\sum_{i=1}^{n} m_i}$$

Let the smallest mean = μ_{11}, $\mu_{ij} = p_{ij}\mu_{11}$ and $\hat{\mu}_{ij} = p_{ij}\hat{\mu}_{11}$

$$FDGSI^{T}_{ij,(r+1)h} = p_{ij}\hat{\mu}_{11} \times DGSI$$

$$= \frac{p_{ij}\left(Y_{11,1h} + Y_{11,2h} + \ldots + Y_{11,rh}\right)}{r\sum_{i=1}^{n} m_i} + \ldots + \frac{p_{ij}\left(Y_{nm_n,1h} + Y_{nm_n,2h} + \ldots + Y_{nm_n,rh}\right)}{rp_{nm_n}\sum_{i=1}^{n} m_i}$$

$$MSEDGSI^{T}_{ij,(r+1)h} = E\left[Y_{ij,(r+1)h} - \frac{p_{ij}\left(Y_{11,1h} + Y_{11,2h} + \ldots + Y_{11,rh}\right)}{r\sum_{i=1}^{n} m_i} - \ldots - \frac{p_{ij}\left(Y_{nm_n,1h} + Y_{nm_n,2h} + \ldots + Y_{nm_n,rh}\right)}{rp_{nm_n}\sum_{i=1}^{n} m_i}\right]^2$$

$$= \left[\mu_{ij} S_{i,h} + \varepsilon_{ij,(r+1)h} - \frac{p_{ij}\left(r\mu_{11} S_{1,h} + \sum_{t=1}^{r} \varepsilon_{11,th} \right)}{r\sum_{i=1}^{n} m_i} - \ldots - \frac{p_{ij}\left(r\mu_{nm_n} S_{n,h} + \sum_{t=1}^{r} \varepsilon_{nm_n,th} \right)}{rp_{nm_n}\sum_{i=1}^{n} m_i} \right]^2$$

$$= E\left[\mu_{ij} S_{i,h} + \varepsilon_{ij,(r+1)h} - \left(\frac{p_{ij}\mu_{11}S_{1,h}}{\sum_{i=1}^{n} m_i} + \ldots + \frac{p_{ij}\mu_{nm_n}S_{n,h}}{p_{nm_n}\sum_{i=1}^{n} m_i} \right) - \left(\frac{p_{ij}\sum_{t=1}^{r}\varepsilon_{11,th}}{r\sum_{i=1}^{n} m_i} + \ldots + \frac{p_{ij}\sum_{t=1}^{r}\varepsilon_{nm_n,th}}{rp_{nm_n}\sum_{i=1}^{n} m_i} \right) \right]^2$$

$$= E\left[\mu_{ij} S_{i,h} + \varepsilon_{ij,(r+1)h} - \left(\frac{\mu_{ij}S_{1,h}}{\sum_{i=1}^{n} m_i} + \ldots + \frac{\mu_{ij}S_{n,h}}{\sum_{i=1}^{n} m_i} \right) - \left(\frac{p_{ij}\sum_{t=1}^{r}\varepsilon_{11,th}}{r\sum_{i=1}^{n} m_i} + \ldots + \frac{p_{ij}\sum_{t=1}^{r}\varepsilon_{nm_n,th}}{rp_{nm_n}\sum_{i=1}^{n} m_i} \right) \right]^2$$

$$= E\left[\mu_{ij} S_{i,h} + \varepsilon_{ij,(r+1)h} - \frac{\mu_{ij}\sum_{i=1}^{n} m_i S_{i,h}}{\sum_{i=1}^{n} m_i} - \left(\frac{p_{ij}\sum_{t=1}^{r}\varepsilon_{11,th}}{r\sum_{i=1}^{n} m_i} + \ldots + \frac{p_{ij}\sum_{t=1}^{r}\varepsilon_{nm_n,th}}{rp_{nm_n}\sum_{i=1}^{n} m_i} \right) \right]^2$$

$$= \mu_{ij}^2 S_{i,h}^2 + \sigma_{ij}^2 + \left(\frac{\mu_{ij}\sum_{i=1}^{n} m_i S_{i,h}}{\sum_{i=1}^{n} m_i} \right)^2 + \frac{p_{ij}^2 r\sigma_{11}^2}{r^2\left(\sum_{i=1}^{n} m_i\right)^2} + \ldots + \frac{p_{ij}^2 r\sigma_{nm_n}^2}{r^2 p_{nm_n}^2\left(\sum_{i=1}^{n} m_i\right)^2}$$

$$- \frac{2\mu_{ij}^2 S_{i,h}\sum_{i=1}^{n} m_i S_{i,h}}{\sum_{i=1}^{n} m_i} + 2\sum_{i=1}^{n}\sum_{k=1}^{m_i-1}\sum_{l=k+1}^{m_i} \frac{p_{ij}^2 r\rho_{ik,il}\sigma_{ik}\sigma_{il}}{r^2\left(\sum_{i=1}^{n} m_i\right)^2 p_{ik}p_{il}} + 2\sum_{i=1}^{n-1}\sum_{u=i+1}^{n}\sum_{k=1}^{m_i}\sum_{v=1}^{m_u} \frac{p_{ij}^2 r\rho_{ik,uv}\sigma_{ik}\sigma_{uv}}{r^2\left(\sum_{i=1}^{n} m_i\right)^2 p_{ik}p_{uv}}$$

$$= \mu_{ij}^2 S_{i,h}^2 + \sigma_{ij}^2 + \left(\frac{\mu_{ij}\sum_{i=1}^{n} m_i S_{i,h}}{\sum_{i=1}^{n} m_i} \right)^2 + \frac{\mu_{ij}^2\sigma_{11}^2}{\mu_{11}^2 r\left(\sum_{i=1}^{n} m_i\right)^2} + \ldots + \frac{\mu_{ij}^2\sigma_{nm_n}^2}{\mu_{nm_n}^2 r\left(\sum_{i=1}^{n} m_i\right)^2} - \frac{2\mu_{ij}^2 S_{i,h}\sum_{i=1}^{n} m_i S_{i,h}}{\sum_{i=1}^{n} m_i}$$

$$+ \frac{2}{r}\sum_{i=1}^{n}\sum_{k=1}^{m_i-1}\sum_{l=k+1}^{m_i} \frac{\mu_{ij}^2\rho_{ik,il}\sigma_{ik}\sigma_{il}}{\mu_{ik}\mu_{il}\left(\sum_{i=1}^{n} m_i\right)^2} + \frac{2}{r}\sum_{i=1}^{n-1}\sum_{u=i+1}^{n}\sum_{k=1}^{m_i}\sum_{v=1}^{m_u} \frac{\mu_{ij}^2\rho_{ik,uv}\sigma_{ik}\sigma_{uv}}{\mu_{ik}\mu_{uv}\left(\sum_{i=1}^{n} m_i\right)^2}$$

Therefore,

$$MSEDGSI_{ij,(r+1)h}^{T} = \mu_{ij}^{2}\left(S_{i,h} - \frac{\sum_{i=1}^{n} m_i S_{i,h}}{\sum_{i=1}^{n} m_i}\right)^2 + \sigma_{ij}^{2} + \frac{\mu_{ij}^{2}}{r\left(\sum_{i=1}^{n} m_i\right)^2}\left(\frac{\sigma_{11}^{2}}{\mu_{11}^{2}} + \ldots + \frac{\sigma_{nm_n}^{2}}{\mu_{nm_n}^{2}}\right)$$
$$+ \frac{2}{r\left(\sum_{i=1}^{n} m_i\right)^2}\sum_{i=1}^{n}\sum_{k=1}^{m_i-1}\sum_{l=k+1}^{m_i}\frac{\mu_{ij}^{2}\rho_{ik,il}\sigma_{ik}\sigma_{il}}{\mu_{ik}\mu_{il}} + \frac{2}{r\left(\sum_{i=1}^{n} m_i\right)^2}\sum_{i=1}^{n-1}\sum_{u=i+1}^{n}\sum_{k=1}^{m_i}\sum_{v=1}^{m_u}\frac{\mu_{ij}^{2}\rho_{ik,uv}\sigma_{ik}\sigma_{uv}}{\mu_{ik}\mu_{uv}} \quad (20)$$

$MSEDGSI_{ij,(r+1)h}^{T} < MSEISI_{ij,(r+1)h}$ if and only if:

$$\mu_{ij}^{2}\left(S_{i,h} - \frac{\sum_{i=1}^{n} m_i S_{i,h}}{\sum_{i=1}^{n} m_i}\right)^2 + \sigma_{ij}^{2} + \frac{\mu_{ij}^{2}}{r\left(\sum_{i=1}^{n} m_i\right)^2}\left(\frac{\sigma_{11}^{2}}{\mu_{11}^{2}} + \ldots + \frac{\sigma_{nm_n}^{2}}{\mu_{nm_n}^{2}}\right) + \frac{2}{r\left(\sum_{i=1}^{n} m_i\right)^2}\sum_{i=1}^{n}\sum_{k=1}^{m_i-1}\sum_{l=k+1}^{m_i}\frac{\mu_{ij}^{2}\rho_{ik,il}\sigma_{ik}\sigma_{il}}{\mu_{ik}\mu_{il}}$$
$$+ \frac{2}{r\left(\sum_{i=1}^{n} m_i\right)^2}\sum_{i=1}^{n-1}\sum_{u=i+1}^{n}\sum_{k=1}^{m_i}\sum_{v=1}^{m_u}\frac{\mu_{ij}^{2}\rho_{ik,uv}\sigma_{ik}\sigma_{uv}}{\mu_{ik}\mu_{uv}} < \left(1 + \frac{1}{r}\right)\sigma_{ij}^{2}$$

$$\left(S_{i,h} - \frac{\sum_{i=1}^{n} m_i S_{i,h}}{\sum_{i=1}^{n} m_i}\right)^2 < \frac{1}{r}\left[\frac{\sigma_{ij}^{2}}{\mu_{ij}^{2}} - \left(\begin{array}{l} \frac{1}{\left(\sum_{i=1}^{n} m_i\right)^2}\left(\frac{\sigma_{11}^{2}}{\mu_{11}^{2}} + \ldots + \frac{\sigma_{nm_n}^{2}}{\mu_{nm_n}^{2}}\right) + \frac{2}{\left(\sum_{i=1}^{n} m_i\right)^2}\sum_{i=1}^{n}\sum_{k=1}^{m_i-1}\sum_{l=k+1}^{m_i}\frac{\rho_{ik,il}\sigma_{ik}\sigma_{il}}{\mu_{ik}\mu_{il}} \\ + \frac{2}{r\left(\sum_{i=1}^{n} m_i\right)^2}\sum_{i=1}^{n-1}\sum_{u=i+1}^{n}\sum_{k=1}^{m_i}\sum_{v=1}^{m_u}\frac{\rho_{ik,uv}\sigma_{ik}\sigma_{uv}}{\mu_{ik}\mu_{uv}} < \left(1 + \frac{1}{r}\right)\sigma_{ij}^{2} \end{array}\right)\right]$$

(21)

$MSEDGSI_{ij,(r+1)h}^{T} < MSEDGSI_{ij,(r+1)h}^{M}$ if and only if:

$$\mu_{ij}^{2}\left(S_{i,h} - \frac{\sum_{i=1}^{n} m_i S_{i,h}}{\sum_{i=1}^{n} m_i}\right)^2 + \sigma_{ij}^{2} + \frac{\mu_{ij}^{2}}{r\left(\sum_{i=1}^{n} m_i\right)^2}\left(\frac{\sigma_{11}^{2}}{\mu_{11}^{2}} + \ldots + \frac{\sigma_{nm_n}^{2}}{\mu_{nm_n}^{2}}\right) + \frac{2}{r\left(\sum_{i=1}^{n} m_i\right)^2}\sum_{i=1}^{n}\sum_{k=1}^{m_i-1}\sum_{l=k+1}^{m_i}\frac{\mu_{ij}^{2}\rho_{ik,il}\sigma_{ik}\sigma_{il}}{\mu_{ik}\mu_{il}}$$
$$+ \frac{2}{r\left(\sum_{i=1}^{n} m_i\right)^2}\sum_{i=1}^{n-1}\sum_{u=i+1}^{n}\sum_{k=1}^{m_i}\sum_{v=1}^{m_u}\frac{\mu_{ij}^{2}\rho_{ik,uv}\sigma_{ik}\sigma_{uv}}{\mu_{ik}\mu_{uv}} < \sigma_{ij}^{2} + \frac{\mu_{ij}^{2}\sigma_{i1}^{2}}{m_i^{2}r\mu_{11}^{2}} + \ldots + \frac{\mu_{ij}^{2}\sigma_{im_i}^{2}}{m_i^{2}r\mu_{im_i}^{2}} + \frac{2\mu_{ij}^{2}}{m_i^{2}r}\sum_{k=1}^{m_i-1}\sum_{l=k+1}^{m_i}\frac{\rho_{ik,il}\sigma_{ik}\sigma_{il}}{\mu_{ik}\mu_{il}}$$

$$\left(S_{i,h}-\frac{\sum_{i=1}^{n}m_i S_{i,h}}{\sum_{i=1}^{n}m_i}\right)^2<\left[\frac{1}{rm_i^2}\left(\frac{\sigma_{i1}^2}{\mu_{11}^2}+\ldots\frac{\sigma_{im_i}^2}{\mu_{im_i}^2}+2\sum_{k=1}^{m_i-1}\sum_{l=k+1}^{m_i}\frac{\rho_{ik,il}\sigma_{ik}\sigma_{il}}{\mu_{ik}\mu_{il}}\right)-\frac{1}{r\left(\sum_{i=1}^{n}m_i\right)^2}\left(\begin{array}{l}\frac{\sigma_{11}^2}{\mu_{11}^2}+\ldots+\frac{\sigma_{nm_n}^2}{\mu_{nm_n}^2}\\+2\sum_{i=1}^{n}\sum_{k=1}^{m_i-1}\sum_{l=k+1}^{m_i}\frac{\rho_{ik,il}\sigma_{ik}\sigma_{il}}{\mu_{ik}\mu_{il}}\\+2\sum_{i=1}^{n-1}\sum_{u=i+1}^{n}\sum_{k=1}^{m_i}\sum_{v=1}^{m_u}\frac{\rho_{ik,uv}\sigma_{ik}\sigma_{uv}}{\mu_{ik}\mu_{uv}}\end{array}\right)\right]$$

(22)

$$MSEWGSI_{ij,(r+1)h}^{M}=\sigma_{ij}^2+\frac{\mu_{ij}^2}{r\mu_i^2}\left[\sigma_1^2+\ldots+\sigma_{m_i}^2+2\sum_{j=1}^{m_i-1}\sum_{l=j+1}^{m_i}\rho_{ij,il}\sigma_{ij}\sigma_{il}\right] \qquad (23)$$

$$WGSI_h^T=\frac{Y_{11,1h}+Y_{12,1h}+\ldots+Y_{nm_i,1h}+\ldots+Y_{nm_n,rh}}{r\hat{\mu}_A}$$

$$=\frac{Y_{11,1h}+Y_{12,1h}+\ldots+Y_{mmi,1h}+\ldots+Y_{nm_n,rh}}{r\left(p_{11}+p_{12}+\ldots+p_{nm_n}\right)\hat{\mu}_{11}}$$

$$FWGSI_{ij,(r+1)h}^T=p_{ij}\hat{\mu}_{11}WGSI_h^T=\frac{p_{ij}\left(Y_{11,1h}+Y_{12,1h}+\ldots+Y_{nmi,1h}+\ldots+Y_{nm_n,rh}\right)}{r\left(p_{11}+p_{12}+\ldots+p_{nm_n}\right)}$$

$$MSEWGSI_{ij,(r+1)h}^T=E\left[\mu_{ij}S_{i,h}+\varepsilon_{ij,(r+1)h}-\frac{p_{ij}\left(Y_{11,1h}+Y_{12,1h}+\ldots+Y_{nmi,1h}+\ldots+Y_{nm_n,rh}\right)}{r\left(p_{11}+p_{12}+\ldots+p_{nm_n}\right)}\right]^2$$

$$=E\left[\mu_{ij}S_{i,h}+\varepsilon_{ij,(r+1)h}-\frac{\mu_{ij}\left(\mu_{11}S_{1,1h}+\mu_{12}S_{1,1h}+\ldots+\mu_{nm_n}S_{n,rh}+\sum_{i=1}^{n}\sum_{j=1}^{m_i}\sum_{t=1}^{r}\varepsilon_{ij,th}\right)}{r\mu_A}\right]^2$$

$$=E\left(\mu_{ij}S_{i,h}+\varepsilon_{ij,(r+1)h}-\frac{\mu_{ij}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mu_{ij}S_{i,h}}{\mu_A}-\frac{\mu_{ij}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\sum_{t=1}^{r}\varepsilon_{ij,th}}{r\mu_A}\right)^2$$

$$=\mu_{ij}^2\left(S_{i,h}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mu_{ij}S_{i,h}}{\mu_A}\right)^2+\sigma_{ij}^2$$

$$+\frac{\mu_{ij}^2}{r\mu_A^2}\left(\sum_{i=1}^{n}\sum_{j=1}^{m_i}\sigma_{ij}^2+2\sum_{i=1}^{n}\sum_{j=1}^{m_i-1}\sum_{u=j+1}^{m_i}\rho_{ij,iu}\sigma_{ij}\sigma_{iu}+2\sum_{i=1}^{n-1}\sum_{v=i+1}^{n}\sum_{j=1}^{m_i}\sum_{l=1}^{m_v}\rho_{ij,vl}\sigma_{ij}\sigma_{vl}\right) \qquad (24)$$

$$MSEWGSI_{ij,(r+1)h}^{T} < MSEISI_{ij,(r+1)h} \text{ if and only if :}$$

$$\mu_{ij}^2\left(S_{i,h}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mu_{ij}S_{i,h}}{\mu_A}\right)^2+\sigma_{ij}^2+\frac{\mu_{ij}^2}{r\mu_A^2}\left(\sum_{i=1}^{n}\sum_{j=1}^{m_i}\sigma_{ij}^2+2\sum_{i=1}^{n}\sum_{j=1}^{m_i-1}\sum_{u=j+1}^{m_i}\rho_{ij,iu}\sigma_{ij}\sigma_{iu}+2\sum_{i=1}^{n-1}\sum_{v=i+1}^{n}\sum_{j=1}^{m_i}\sum_{l=1}^{m_v}\rho_{ij,vl}\sigma_{ij}\sigma_{vl}\right)$$

$$<\left(1+\frac{1}{r}\right)\sigma_{ij}^2$$

$$\mu_{ij}^2\left(S_{i,h}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mu_{ij}S_{i,h}}{\mu_A}\right)^2+\frac{\mu_{ij}^2}{r\mu_A^2}\sigma_A^2<\frac{1}{r}\sigma_{ij}^2$$

$$\left(S_{i,h}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mu_{ij}S_{i,h}}{\mu_A}\right)^2<\frac{1}{r}\left(\frac{\sigma_{ij}^2}{\mu_{ij}^2}-\frac{\sigma_A^2}{\mu_A^2}\right) \tag{25}$$

$$MSEWGSI_{ij,(r+1)h}^{T} < MSEWGSI_{ij,(r+1)}^{M} \text{ if and only if :}$$

$$\mu_{ij}^2\left(S_{i,h}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mu_{ij}S_{i,h}}{\mu_A}\right)^2+\sigma_{ij}^2+\frac{\mu_{ij}^2}{r\mu_A^2}\left(\sum_{i=1}^{n}\sum_{j=1}^{m_i}\sigma_{ij}^2+2\sum_{i=1}^{n}\sum_{j=1}^{m_i-1}\sum_{u=j+1}^{m_i}\rho_{ij,iu}\sigma_{ij}\sigma_{iu}+2\sum_{i=1}^{n-1}\sum_{v=i+1}^{n}\sum_{j=1}^{m_i}\sum_{l=1}^{m_v}\rho_{ij,vl}\sigma_{ij}\sigma_{vl}\right)$$

$$<\sigma_{ij}^2+\frac{\mu_{ij}^2}{r\mu_i^2}\left[\sigma_1^2+\ldots+\sigma_{m_i}^2+2\sum_{j=1}^{m_i-1}\sum_{l=j+1}^{m_i}\rho_{ij,il}\sigma_{ij}\sigma_{il}\right]$$

$$\left(S_{i,h}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{m_i}\mu_{ij}S_{i,h}}{\mu_A}\right)^2<\frac{1}{r}\left(\frac{\sigma_i^2}{\mu_i^2}-\frac{\sigma_A^2}{\mu_A^2}\right) \tag{26}$$

Appendix 3
A special case: two levels
The Additive Model

The additive model is specified as:

$$Y_{i,th}=\mu_i+S_{i,h}+\varepsilon_{i,th} \tag{27}$$

Forecast for item i, the hth season in year r+1 using ISI is:

$$FISI_{i,(r+1)h} = \mu_i + ISI_{i,h} = \frac{1}{r}\sum_{t=1}^{r} Y_{i,th} \qquad (28)$$

MSE of the forecast is:

$$MSEISI_i = E\left(Y_{i,(r+1)h} - \frac{1}{r}\sum_{t=1}^{r} Y_{i,th}\right)^2$$

$$= E\left[\mu_i + S_h + \varepsilon_{i,(r+1)h} - \mu_i - S_h - \frac{1}{r}\left(\varepsilon_{i,1h} + \varepsilon_{i,2h} + \ldots + \varepsilon_{i,rh}\right)\right]^2$$

$$= \sigma_i^2 + \frac{\sigma_i^2}{r} \qquad (29)$$

Suppose we can find m items with similar seasonal patterns. It might be better to estimate their seasonality from the group.

The GSI estimator is given as:

$$GSI_h = \frac{1}{mr}\sum_{i=1}^{m}\sum_{t=1}^{r} Y_{i,th} - \frac{1}{mqr}\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{h=1}^{q} Y_{i,th}$$

$$\hat{\mu}_i = \frac{1}{qr}\sum_{t=1}^{r}\sum_{h=1}^{q} Y_{i,th}$$

Therefore, the forecast of item i, the hth season in year r+1 is:

$$FGSI_{i,(r+1)h} = \hat{\mu}_i + GSI_h$$

$$= \frac{1}{qr}\sum_{t=1}^{r}\sum_{h=1}^{q} Y_{i,th} + \frac{1}{mr}\sum_{i=1}^{m}\sum_{t=1}^{r} Y_{i,th} - \frac{1}{mqr}\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{h=1}^{q} Y_{i,th}$$

MSE of the forecast is as follows:

$$MSEGSI_i = E\left(Y_{i,(r+1)h} - \frac{1}{qr}\sum_{t=1}^{r}\sum_{h=1}^{q} Y_{i,th} - \frac{1}{mr}\sum_{i=1}^{m}\sum_{t=1}^{r} Y_{i,th} + \frac{1}{mqr}\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{h=1}^{q} Y_{i,th}\right)^2$$

$$= E\begin{pmatrix} \mu_i + S_{i,h} + \varepsilon_{i,(r+1)h} - \mu_i - \frac{1}{qr}\sum_{t=1}^{r}\sum_{h=1}^{q}\varepsilon_{i,th} - \frac{\mu_A}{m} - \frac{1}{m}\sum_{i=1}^{m} S_{i,h} \\ -\frac{1}{mr}\sum_{i=1}^{m}\sum_{t=1}^{r}\varepsilon_{i,th} + \frac{\mu_A}{m} + \frac{1}{mqr}\sum_{i=1}^{m}\sum_{t=1}^{r}\sum_{h=1}^{q}\varepsilon_{i,th} \end{pmatrix}^2$$

$$= \left(S_{i,h} - \frac{1}{m}\sum_{i=1}^{m} S_{i,h} \right)^2 + \sigma_i^2 + \frac{\sigma_i^2}{qr} + \frac{\sigma_A^2}{m^2 r} + \frac{\sigma_A^2}{m^2 qr} + \frac{2r\left(\sigma_i^2 + \sum_{j \neq i} \rho_{ij}\sigma_i\sigma_j \right)}{mqr^2} - \frac{2\left(qr\sigma_i^2 + qr\sum_{j \neq i} \rho_{ij}\sigma_i\sigma_j \right)}{mq^2 r^2}$$

$$- \frac{2r\left(\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_m^2 + 2\sum_{j=1}^{m-1}\sum_{l=j+1}^{m} \rho_{jl}\sigma_j\sigma_l \right)}{m^2 qr^2}$$

$$= \left(S_{i,h} - \frac{1}{m}\sum_{i=1}^{m} S_{i,h} \right)^2 + \sigma_i^2 + \frac{\sigma_i^2}{qr} + \frac{\sigma_A^2}{m^2 r} - \frac{\sigma_A^2}{m^2 qr}$$

$$= \left(S_{i,h} - \frac{1}{m}\sum_{i=1}^{m} S_{i,h} \right)^2 + \left(1 + \frac{1}{qr} \right)\sigma_i^2 + \frac{(q-1)\sigma_A^2}{m^2 qr} \qquad (30)$$

$MSEGSI_i < MSEISI_i$ if and only if

$$\left(S_{i,h} - \frac{1}{m}\sum_{i=1}^{m} S_{i,h} \right)^2 + \left(1 + \frac{1}{qr} \right)\sigma_i^2 + \frac{(q-1)\sigma_A^2}{m^2 qr} < \sigma_i^2 + \frac{\sigma_i^2}{r}$$

$$\left(S_{i,h} - \frac{1}{m}\sum_{i=1}^{m} S_{i,h} \right)^2 < \frac{q-1}{qr}\left(\sigma_i^2 - \frac{\sigma_A^2}{m^2} \right) \qquad (31)$$

The Mixed Model

The mixed model is specified as:

$$Y_{i,th} = \mu_i S_{i,h} + \varepsilon_{i,th} \qquad (32)$$

MSE using ISI is the same as in the additive model.

$$DGSI_h = \frac{ISI_{1,h} + ISI_{2,h} + \ldots + ISI_{m,h}}{m}$$

$$= \frac{\frac{Y_{1,1h} + Y_{1,2h} + \ldots + Y_{1,rh}}{r\hat{\mu}_1} + \ldots + \frac{Y_{m,1h} + Y_{m,2h} + \ldots + Y_{m,rh}}{r\hat{\mu}_m}}{m}$$

The same forecast using DGSI is :

$$FDGSI_{i,(r+1)h} = p_i \hat{\mu}_1 * DGSI_h$$

$$= \frac{p_i(Y_{1,1h} + Y_{1,2h} + \ldots + Y_{1,rh})}{mr} + \frac{p_i(Y_{2,1h} + Y_{2,2h} + \ldots + Y_{2,rh})}{mrp_2} + \ldots + \frac{p_i(Y_{m,1h} + Y_{m,2h} + \ldots + Y_{m,rh})}{mrp_m}$$

Mean square error is :

$$MSEDGSI_i = E\left(Y_{i,(r+1)h} - \frac{p_i(Y_{1,1h}+Y_{1,2h}+...+Y_{1,rh})}{mrp_1} - ... - \frac{p_i(Y_{m,1h}+Y_{m,2h}+...+Y_{m,rh})}{mrp_m} \right)^2$$

$$= E\left[\mu_i S_{i,h} + \varepsilon_{i,(r+1)h} - \frac{\mu_i \sum_{i=1}^{m} S_{i,h}}{m} - \frac{p_i(\varepsilon_{1,1h}+\varepsilon_{1,2h}+...+\varepsilon_{1,rh})}{mr} - ... - \frac{p_i(\varepsilon_{m,1h}+\varepsilon_{m,2h}+...+\varepsilon_{m,rh})}{mrp_m} \right]^2$$

$$= \mu_i^2 \left(S_{i,h} - \frac{1}{m}\sum_{i=1}^{m} S_{i,h} \right)^2 + \sigma_i^2 + \frac{p_i^2 r\sigma_1^2}{m^2r^2} + ... + \frac{p_i^2 r\sigma_m^2}{m^2r^2p_m^2} + 2\sum_{j=1}^{m-1}\sum_{l=j+1}^{m} \frac{p_i}{mrp_j}\frac{p_i}{mrp_l} r\rho_{jl}\sigma_j\sigma_l$$

$$= \mu_i^2 \left(S_{i,h} - \frac{1}{m}\sum_{i=1}^{m} S_{i,h} \right)^2 + \sigma_i^2 + \frac{p_i^2 r\sigma_1^2}{m^2r^2} + ... + \frac{p_i^2 r\sigma_m^2}{m^2r^2p_m^2} + 2\frac{p_i^2}{m^2r}\sum_{j=1}^{m-1}\sum_{l=j+1}^{m} \frac{1}{p_j}\frac{1}{p_l} \rho_{jl}\sigma_j\sigma_l$$

$$= \mu_i^2 \left(S_{i,h} - \frac{1}{m}\sum_{i=1}^{m} S_{i,h} \right)^2 + \sigma_i^2 + \frac{p_i^2 \sigma_1^2}{m^2r} + ... + \frac{p_i^2 \sigma_m^2}{m^2rp_m^2} + 2\frac{p_i^2}{m^2r}\sum_{j=1}^{m-1}\sum_{l=j+1}^{m} \frac{1}{p_j}\frac{1}{p_l} \rho_{jl}\sigma_j\sigma_l$$

$$= \mu_i^2 \left(S_{i,h} - \frac{1}{m}\sum_{i=1}^{m} S_{i,h} \right)^2 + \sigma_i^2 + \frac{\mu_i^2 \sigma_1^2}{m^2r\mu_1^2} + ... + \frac{\mu_i^2 \sigma_m^2}{m^2r\mu_m^2} + 2\frac{\mu_i^2}{m^2r}\sum_{j=1}^{m-1}\sum_{l=j+1}^{m} \frac{1}{\mu_j}\frac{1}{\mu_l} \rho_{jl}\sigma_j\sigma_l \quad (33)$$

$MSEDGSI_i < MSEISI_i$ if and only if

$$\mu_i^2 \left(S_{i,h} - \frac{1}{m}\sum_{i=1}^{m} S_{i,h} \right)^2 + \sigma_i^2 + \frac{\mu_i^2 \sigma_1^2}{m^2r\mu_1^2} + ... + \frac{\mu_i^2 \sigma_m^2}{m^2r\mu_m^2} + 2\frac{\mu_i^2}{m^2r}\sum_{j=1}^{m-1}\sum_{l=j+1}^{m} \frac{1}{\mu_j}\frac{1}{\mu_l} \rho_{jl}\sigma_j\sigma_l < \sigma_i^2 + \frac{\sigma_i^2}{r}$$

$$\mu_i^2 \left(S_{i,h} - \frac{1}{m}\sum_{i=1}^{m} S_{i,h} \right)^2 < \frac{\sigma_i^2}{r} - \left(\frac{\mu_i^2 \sigma_1^2}{m^2r\mu_1^2} + ... + \frac{\mu_i^2 \sigma_m^2}{m^2r\mu_m^2} \right) - 2\frac{\mu_i^2}{m^2r}\sum_{j=1}^{m-1}\sum_{l=j+1}^{m} \frac{1}{\mu_j}\frac{1}{\mu_l} \rho_{jl}\sigma_j\sigma_l$$

$$\left(S_{i,h} - \frac{1}{m}\sum_{i=1}^{m} S_{i,h} \right)^2 < \frac{1}{r}\left[\frac{\sigma_i^2}{\mu_i^2} - \frac{1}{m^2}\left(\sum_{i=1}^{m} \frac{\sigma_i^2}{\mu_i^2} + 2\sum_{j=1}^{m-1}\sum_{l=j+1}^{m} \frac{1}{\mu_j}\frac{1}{\mu_l} \rho_{jl}\sigma_j\sigma_l \right) \right] \quad (34)$$

$$WGSI_h = \frac{\frac{Y_{1,1h}+Y_{2,1h}+...+Y_{m,1h}}{\hat{\mu}_A} + \frac{Y_{1,2h}+Y_{2,2h}+...+Y_{m,2h}}{\hat{\mu}_A} + ... + \frac{Y_{1,rh}+Y_{2,rh}+...+Y_{m,rh}}{\hat{\mu}_A}}{r}$$

$$= \frac{Y_{1,1h}+...+Y_{1,rh}+...+Y_{m,1h}+...+Y_{m,rh}}{r(p_1+p_2+...+p_m)\hat{\mu}_1}$$

The forecast of hth season in year $r+1$ for the ith item using WGSI is :

$$FWGSI_{i,(r+1)h} = p_i\hat{\mu}_1 * WGSI_h$$

$$= \frac{p_i(Y_{1,1h}+Y_{1,2h}+...+Y_{1,rh}+...Y_{m,1h}+Y_{m,2h}+...+Y_{m,rh})}{r(p_1+p_2+...+p_m)}$$

Mean square error is :

$$MSEWGSI_i = E\left(Y_{i,(r+1)h} - \frac{p_i(Y_{1,1h} + Y_{1,2h} + \ldots + Y_{1,rh} + \ldots + Y_{m,1h} + Y_{m,2h} + \ldots + Y_{m,rh}}{r(p_1 + p_2 + \ldots + p_m)}\right)^2$$

$$= E\left[\mu_i S_{i,h} + \varepsilon_{i,(r+1)h} - \frac{\mu_i \sum_{i=1}^{m} \mu_i S_{i,h}}{\mu_A} - \frac{p_i(\varepsilon_{1,1h} + \ldots + \varepsilon_{1,rh} + \ldots + \varepsilon_{m,1h} + \ldots + \varepsilon_{m,rh})}{r(p_1 + p_2 + \ldots + p_m)}\right]^2$$

$$= \mu_i^2 \left(S_{i,h} - \frac{\sum_{i=1}^{m} \mu_i S_{i,h}}{\mu_A}\right)^2 + \sigma_i^2 + \frac{p_i^2}{r^2(p_1 + p_2 + \ldots + p_m)^2} E\left[\varepsilon_{1,1h} + \ldots + \varepsilon_{1,rh} + \ldots + \varepsilon_{m,1h} + \ldots + \varepsilon_{m,rh}\right]^2$$

$$= \mu_i^2 \left(S_{i,h} - \frac{\sum_{i=1}^{m} \mu_i S_{i,h}}{\mu_A}\right)^2 + \sigma_i^2 + \frac{p_i^2}{r^2(p_1 + p_2 + \ldots + p_m)^2}\left[r\sigma_1^2 + \ldots + r\sigma_m^2 + 2\sum_{j=1}^{m-1}\sum_{l=j+1}^{m} r\rho_{jl}\sigma_j\sigma_l\right]$$

$$= \mu_i^2 \left(S_{i,h} - \frac{\sum_{i=1}^{m} \mu_i S_{i,h}}{\mu_A}\right)^2 + \sigma_i^2 + \frac{\mu_i^2}{r\mu_A^2}\left[\sigma_1^2 + \ldots + \sigma_m^2 + 2\sum_{j=1}^{m-1}\sum_{l=j+1}^{m} \rho_{jl}\sigma_j\sigma_l\right] \qquad (35)$$

$MSEWGSI_i < MSEISI_i$ if and only if

$$\mu_i^2 \left(S_{i,h} - \frac{\sum_{i=1}^{m} \mu_i S_{i,h}}{\mu_A}\right)^2 + \sigma_i^2 + \frac{\mu_i^2}{r\mu_A^2}\left[\sigma_1^2 + \ldots + \sigma_m^2 + 2\sum_{j=1}^{m-1}\sum_{l=j+1}^{m} \rho_{jl}\sigma_j\sigma_l\right] < \sigma_i^2 + \frac{\sigma_i^2}{r}$$

$$\mu_i^2 \left(S_{i,h} - \frac{\sum_{i=1}^{m} \mu_i S_{i,h}}{\mu_A}\right)^2 < \frac{\sigma_i^2}{r} - \frac{\mu_i^2}{r\mu_A^2}\left[\sigma_1^2 + \ldots + \sigma_m^2 + 2\sum_{j=1}^{m-1}\sum_{l=j+1}^{m} \rho_{jl}\sigma_j\sigma_l\right]$$

$$\left(S_{i,h} - \frac{\sum_{i=1}^{m} \mu_i S_{i,h}}{\mu_A}\right)^2 < \frac{1}{r}\left(\frac{\sigma_i^2}{\mu_i^2} - \frac{\sigma_1^2 + \ldots + \sigma_m^2 + 2\sum_{j=1}^{m-1}\sum_{l=j+1}^{m} \rho_{jl}\sigma_j\sigma_l}{\mu_A^2}\right) \qquad (36)$$